MATH241 Exam 3 Fall 2021 (Justin Wyss-Gallifent)

Solutions

1. The following integral can be evaluated without integrating but by figuring out the volume of the shape. Describe the shape and give the numerical value of the integral. [5 pts]

$$\int_{-2}^{2} \int_{1}^{7} \sqrt{4-x^2}\, dy\, dx$$

Solution:

This is the volume of a half-cylinder of radius 2 with length 6. Thus the volume is:

$$\frac{1}{2}\pi 2^2 \cdot 6$$

2. Evaluate the following iterated double integral: [10 pts]

$$\int_{0}^{1} \int_{0}^{y} 4x - y\, dx\, dy$$

You Should Evaluate This Integral!

Solution:

We have:

$$\begin{aligned}
\int_{0}^{1} \int_{0}^{y} 4x - y\, dx\, dy &= \int_{0}^{1} 2x^2 - xy\Big|_0^y\, dy \\
&= \int_0^1 2y^2 - y^2\, dy \\
&= \int_0^1 y^2\, dy \\
&= \frac{1}{3}y^3\Big|_0^1 \\
&= \frac{1}{3}
\end{aligned}$$

3. Let R be the 2D region between the graphs of $y = 5 - x^2$ and $y = 2x$. Write down the iterated double integral for $\iint_R xy\,dA$, treating R as vertically simple. [10 pts]

You Should Not Evaluate Your Resulting Integral!

Solution:

The graphs meet at:

$$\begin{aligned} 5 - x^2 &= 2x \\ x^2 + 2x - 5 &= 0 \\ x &= \frac{-2 \pm \sqrt{2^2 - 4(1)(-5)}}{2} \\ &= \frac{-2 \pm \sqrt{24}}{2} \\ &= \frac{-2 \pm \sqrt{2}\sqrt{6}}{2} \\ &= -1 \pm \sqrt{6} \end{aligned}$$

So the integral is:

$$\iint_R xy\,dA = \int_{-1-\sqrt{6}}^{-1+\sqrt{6}} \int_{2x}^{5-x^2} xy\,dy\,dx$$

4. Write down a parametrization $\mathbf{r}(?,?) = ...$ of the part of the cylinder $x^2 + y^2 = 9$ between $z = 1$ and $z = 3$ and which is NOT in the first octant. [5 pts]

Solution:

The clearest is:

$$\mathbf{r}(\theta, z) = 3\cos\theta\hat{\boldsymbol{\imath}} + 3\sin\theta\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}$$
$$\frac{\pi}{2} \le \theta \le 2\pi$$
$$1 \le z \le 3$$

5. Let D be the 3D solid object in the first octant and bounded by the planes $y = 4$, $y = x$, $y = 2x$, and $x + y + z = 10$. Write down an iterated triple integral in rectangular coordinates for the volume of D. [15 pts]

You Should Not Evaluate Your Resulting Integral!

Solution:

The corresponding region in the xy-plane is horizontally simple between $y = 0$ and $y = 4$ with $x = \frac{1}{2}y$ on the left and $x = y$ on the right.

Thus we have:

$$\int_0^4 \int_{\frac{1}{2}y}^{y} \int_0^{10-x-y} 1\,dz\,dy\,dx$$

6. Let D be the 3D solid object above the xy-plane, inside the cylinder $x^2+y^2=4$ and below/outside the cone $z=\sqrt{\frac{x^2+y^2}{3}}$. If the density of D at the point (x,y,z) equals x^2z, write down the iterated triple integral in spherical coordinates for the mass of D. [20 pts]

You Should Not Evaluate Your Resulting Integral!

Solution:

The cone is $\phi=\pi/3$ and the cylinder is $\rho^2\sin^2\phi=4$ or $\rho=2\csc\phi$. The integral is then:

$$\int_0^{2\pi}\int_{\pi/3}^{\pi/2}\int_0^{2\csc\phi}(\rho\sin\phi\cos\theta)^2(\rho\cos\phi)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

7. Convert the following integral to an iterated double integral in polar coordinates. [15 pts]

$$\int_1^2 \int_{-\sqrt{1-(x-1)^2}}^{+\sqrt{1-(x-1)^2}} x\, dy\, dx$$

You Should Not Evaluate Your Resulting Integral!

Solution:

The region is inside the right-half of the circle $(x-1)^2 + y^2 = 1$, or $r = 2\cos\theta$. The far function is then $r = 2\cos\theta$ and the near function is $x = 1$, or $r\cos\theta = 1$ or $r = \sec\theta$. These meet at $\theta = \pm\pi/4$.

Thus we have:

$$\int_1^2 \int_{-\sqrt{1-(x-1)^2}}^{+\sqrt{1-(x-1)^2}} x\, dy\, dx = \int_{-\pi/4}^{\pi/4} \int_{\sec\theta}^{2\cos\theta} r\cos\theta(r)\, dr\, d\theta$$

8. Let R be the 2D region bounded by the three lines: [20 pts]

$$y = x + 1$$

$$y = 3x$$

$$y = 4x$$

Use the change of variables $x = u + v$ and $y = u + 4v$ to convert the following integral to a double iterated integral in the uv-plane.

$$\iint_R y\, dA$$

You Should Not Evaluate Your Resulting Integral!

Solution:

The three lines change as follows:

$y = x + 1 \Rightarrow u + 4v = u + v + 1 \Rightarrow 3v = 1 \Rightarrow v = 1/3$

$y = 3x \Rightarrow u + 4v = 3(u + v) \Rightarrow v = 2u$

$y = 4x \Rightarrow u + 4v = 4(u + v) \Rightarrow u = 0$

Thus the new region S is bounded by these lines. Note that the lines $v = 1/3$ and $v = 2u$ meet when $2u = 1/3$ so $u = 1/6$.

The Jacobian is:

$$\begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} = 3$$

Hence we have:

$$\begin{aligned} \iint_R x\, dR &= \iint_S (u + 4v)|3|\, dA \\ &= 3 \int_0^{1/6} \int_{2u}^{1/3} u + 4v\, dv\, du \end{aligned}$$