MATH241 Fall 2023 Exam 3 (Justin W-G) Solutions

Name (Neatly):	
UID (Neatly):	

Instructions:

1. Please do all problems on the pages and in the spaces provided. This exam will be scanned into Gradescope and if your answers are not in the correct locations they will not be found or graded!
2. Only simplify Calculus 3 related calculations unless otherwise specified.

1. Write TRUE or FALSE in the box to the right. No justification is required. Unreadable or ambiguous answers will be marked as incorrect. [10 pts]

Solution:

Statement	TRUE/FALSE
Every vertically simple region is also horizontally simple.	F
In spherical, $x = \rho \sin\phi \cos\theta$.	T
We can never use a double integral to find volume.	F
$\iint_R 1\, dA$ yields volume.	T/F
The parameterization of a surface requires two variables.	T

2. The following integral can be evaluated without integrating but by understanding something about the shape and integrand:

$$\int_0^4 \int_0^{4-x} 3\, dy\, dx$$

(a) Describe the shape as best you can. Either words or a picture are acceptable. [5 pts]

Solution:

This is wedge-shaped. It is essentially a triangle in the xy-plane which extends from $z = 0$ to $z = 3$.

(b) Give the simplified numerical value for the integral. [5 pts]

Solution:

The integral is the volume of this shape:

$$\frac{1}{2}(4)(4)(3) = 24$$

3. Given the following iterated integral:

$$\int_0^{\pi/2} \int_0^{2\cos\theta} 2\, dr\, d\theta$$

(a) Draw the region R corresponding to the following iterated integral. All you need to do is draw the region. [5 pts]

Solution:

I have not typeset this. It's the top half of the disk $(x-1)^2 + y^2 \le 1$.

(b) Evaluate the integral. [10 pts]

Solution:

We have:

$$\begin{aligned}
\int_0^{\pi/2} \int_0^{2\cos\theta} 2\, dr\, d\theta &= \int_0^{\pi/2} 2r\Big|_0^{2\cos\theta} d\theta \\
&= \int_0^{\pi/2} 4\cos\theta\, d\theta \\
&= 4\sin\theta\Big|_0^{\pi/2} \\
&= 4
\end{aligned}$$

4. Suppose R is the region above $y = x^2$, below $y = 9$, and in the first quadrant. Set up the iterated double integral in rectangular coordinates treating R as horizontally simple for: [10 pts]

$$\iint_R xy\,dA$$

DO NOT EVALUATE!

Note: A picture is not necessary but can earn partial credit.

Solution:

The result is:

$$\iint_R xy\,dA = \int_0^9 \int_0^{\sqrt{y}} xy\,dx\,dy$$

5. Suppose R is the region inside $r = 3\cos\theta$ and outside $r = 1 + \cos\theta$. Set up the iterated double integral in polar coordinates for: [10 pts]

$$\iint_R x\,dA$$

DO NOT EVALUATE!

Note: A picture is not necessary but can earn partial credit.

Solution:

The functions meet when:

$$\begin{aligned} 3\cos\theta &= 1 + \cos\theta \\ \cos\theta &= \frac{1}{2} \\ \theta &= \pm\pi/3 \end{aligned}$$

Thus:

$$\iint_R x\,dA = \int_{-\pi/3}^{\pi/3} \int_{1+\cos\theta}^{3\cos\theta} (r\cos\theta) r\,dr\,d\theta$$

6. Suppose D is the solid between $r = \sin\theta$ and $r = 2\sin\theta$ and between $z = 1$ and $z = x^2 + y^2$. [15 pts]
Write down the iterated triple integral in cylindrical coordinates for the volume of D.

DO NOT EVALUATE!

Note: Pictures are not necessary but can earn partial credit.

Solution:

The result is:

$$\iiint_D 1\,dV = \int_0^{\pi} \int_{\sin\theta}^{2\sin\theta} \int_1^{3+r^2} r\,dz\,dr\,d\theta$$

7. Suppose D is the solid inside $z = \sqrt{3x^2 + 3y^2}$, below $z = 10$ and outside $x^2 + y^2 + z^2 = 2$. If the density is given by $f(x, y, z) = z^4$, write down the iterated triple integral in spherical coordinates for the mass of D. [15 pts]

DO NOT EVALUATE!

Note: Pictures are not necessary but can earn partial credit.

Solution:

The result is:

$$\iiint_D z^4 \, dV = \int_0^{2\pi} \int_0^{\pi/6} \int_{\sqrt{2}}^{10 \sec \phi} (\rho \cos \phi)^4 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

8. Write down a parameterization of the part of the plane $y = 4 - x$ in the first octant and between $z = 0$ and $z = 7$. No sketch is required. [5 pts]

Solution:

One such parameterization is:

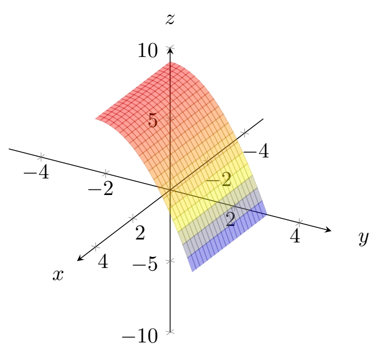
$$\bar{\boldsymbol{r}}(x, z) = x\,\hat{\boldsymbol{\imath}} + (4 - x)\,\hat{\boldsymbol{\jmath}} + z\,\hat{\boldsymbol{k}} 0 \leq x \leq 4 0 \leq z \leq 7$$

9. Sketch the surface parameterized by: [5 pts]

$$\begin{gathered}\bar{\boldsymbol{r}}(x, y) = x\,\hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}} + (9 - y^2)\,\hat{\boldsymbol{k}} \\ 0 \leq x \leq 4 \\ 0 \leq y \leq 3\end{gathered}$$

Solution:

This is a part of the parabolic sheet in the first octant between $x = 0$ and $x = 4$.



10. Let R be the region bounded by the lines $y = 1$, $y = \frac{1}{4}x$, and $x - 3y = 2$. Consider the integral: [15 pts]

$$\iint_R \frac{y}{x-3y}\, dA$$

Use the substitution $x = 3u + v$ and $y = u$ to convert this integral to an iterated integral in the uv-plane.

DO NOT EVALUATE!

Note: Pictures are not necessary but can earn partial credit.

Solution:

We convert the regions:

$y = 1 \Longrightarrow u = 1$

$y = \frac{1}{4}x \Longrightarrow u = \frac{1}{4}(3u + v) \Longrightarrow v = u$

$x - 3y = 2 \Longrightarrow 3u + v - 3u = 2 \Longrightarrow v = 2$

The Jacobian is:

$$\begin{vmatrix} 3 & 1 \\ 1 & 0 \end{vmatrix} = -1$$

Then we have:

$$\iint_R \frac{y}{x-3y}\, dA = \iint_S \frac{u}{3u+v-3u} |-1|\, dA = \int_1^2 \int_u^2 \frac{u}{v}\, dv\, du$$