Math 241 Exam 3 Spring 2020 Solutions **Justin Wyss-Gallifent**

1. Consider the integrals: [25pts]

$$A = \int_0^1 \int_0^x x^2 y \, dy \, dx \qquad \text{and} \qquad B = \int_0^2 \int_0^x x^2 y \, dy \, dx$$

(a) Either $A < B$, $A = B$ or $A > B$. Without calculating either integral explain in a few sentences which is true and why. You may use pictures too if you feel it helps.

Solution: Since the function is positive we are finding the volume under it and since R (the base) for the integral B is larger than for the integral A we know that the volume for B is larger. Thus $B > A$.

(b) Calculate both integrals. Were you correct?

Solution:

We have:

$$A = \int_0^1 \int_0^x x^2 y \, dy \, dx = \int_0^1 \frac{1}{2} x^2 y^2 \bigg|_0^x \, dx = \int_0^1 \frac{1}{2} x^4 \, dx = \frac{1}{10} x^5 \bigg|_0^1 = \frac{1}{10}$$

$$B = \int_0^2 \int_0^x x^2 y \, dy \, dx = \int_0^2 \frac{1}{2} x^2 y^2 \bigg|_0^x \, dx = \int_0^2 \frac{1}{2} x^4 \, dx = \frac{1}{10} x^5 \bigg|_0^2 = \frac{32}{10}$$

So yes, I was right!

2. Let R be the region inside $r = 2\cos\theta$ and to the right of $x = 1$. Consider the integral: [25pts]

$$\iint_R x\, dA$$

(a) Draw a picture of R.

Solution: Omitted here but it's the right half of a circle of radius 1 centered at $(1, 0)$.

(b) Parametrize R as polar and write down the corresponding iterated integral. Do not evaluate.

Solution: We have:

$$\iint_R x\, dA = \int_{-\pi/4}^{\pi/4} \int_{\sec\theta}^{2\cos\theta} (r\cos\theta) r\, dr\, d\theta$$

(c) Parametrize R as vertically simple and write down the corresponding iterated integral. Do not evaluate.

Solution: We have:

$$\iint_R x\, dA = \int_1^2 \int_{-\sqrt{1-(x-1)^2}}^{+\sqrt{1-(x-1)^2}} x\, dy\, dx$$

(d) Parametrize R as horizontally simple and write down the corresponding iterated integral. Do not evaluate.

Solution: We have:

$$\iint_R x\, dA = \int_{-1}^{1} \int_1^{1+\sqrt{1-y^2}} x\, dy\, dx$$

3. Let D be the solid above the cone $\phi = \phi_0$ and inside the sphere of radius $\rho = \rho_0$. Here both ϕ_0 and ρ_0 are unknown constants. [25pts]

(a) Use a triple integral in spherical coordinates to find a formula for the volume of D. Your answer will have ϕ_0 and ρ_0 in it.

Solution: The volume is:

$$\begin{aligned}
\iiint_D 1\,dV &= \int_0^{2\pi}\int_0^{\phi_0}\int_0^{\rho_0} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \\
&= \int_0^{2\pi}\int_0^{\phi_0} \frac{1}{3}\rho^3 \sin\phi \Big|_0^{\rho_0} \, d\phi \, d\theta \\
&= \int_0^{2\pi}\int_0^{\phi_0} \frac{1}{3}\rho_0^3 \sin\phi \, d\phi \, d\theta \\
&= \int_0^{2\pi} -\frac{1}{3}\rho_0^3 \cos\phi \Big|_0^{\phi_0} \, d\theta \\
&= \int_0^{2\pi} \frac{1}{3}\rho_0^3 \left(-\cos\phi_0 + 1\right) \, d\theta \\
&= \frac{1}{3}\rho_0^3 \left(1 - \cos\phi_0\right) \theta \Big|_0^{2\pi} \\
&= \frac{2\pi}{3}\rho_0^3 \left(1 - \cos\phi_0\right)
\end{aligned}$$

(b) When $\phi_0 = \pi/2$ and $\rho_0 = 3$ what does D look like? When you plug these into your formula do you get the answer you expect? Explain in a sentence or two.

Solution: In this case D is a hemisphere of radius 3 so we would expect to get:

$$\frac{1}{2}\left[\frac{4}{3}\pi(3)^3\right] = 18\pi$$

If we plug the values into the formula we get:

$$\frac{2\pi}{3}(3)^3(1-0) = 18\pi$$

So they match.

(c) When $\phi_0 = \pi$ and $\rho_0 = 5$ what does D look like? When you plug these into your formula do you get the answer you expect? Explain in a sentence or two.

Solution:

In this case D is a sphere of radius 5 so we would expect to get:

$$\frac{4}{3}\pi(5)^3 = \frac{500\pi}{3}$$

If we plug the values into the formula we get:

$$\frac{2\pi}{3}(5)^3(1-(-1)) = \frac{500\pi}{3}$$

So they match.

(d) Explain in a few sentences why cylindrical coordinates would be a really difficult way to do part (a).

Solution: There are several different ways this could be explained and anything reasonable is fine. One thing is that in cylindrical coordinates we'd need the radius of R, which arises from where the sphere meets the cone, and this would take some work. Moreover when $\phi_0 > \pi$ the sense of "top" and "bottom" functions is much more confusing and several separate integrals would be required.

4. Let R be the region bounded by the lines $y = x$, $y = x - 4$, $y = -x$ and $y = -x + 4$. Consider the integral: [25pts]

$$\iint_R x\, dA$$

(a) Parametrize R using two vertically simple regions and evaluate.

Solution: The top and bottom functions both change at $x = 2$ and so we need to split up the integral there. The result is:

$$\begin{aligned}
\iint_R x\, dA &= \int_0^2 \int_{-x}^{x} x\, dy\, dx + \int_2^4 \int_{x-4}^{4-x} x\, dy\, dx \\
&= \int_0^2 xy\Big|_{-x}^{x} dx + \int_2^4 xy\Big|_{x-4}^{4-x} dx \\
&= \int_0^2 2x^2\, dx + \int_2^4 -2x^2 + 8x\, dx \\
&= \frac{2}{3}x^3\Big|_0^2 + \left(-\frac{2}{3}x^3 + 4x^2\right)\Big|_2^4 \\
&= \frac{2}{3}(2)^3 + \left(-\frac{2}{3}(4)^3 + 4(4)^2\right) - \left(-\frac{2}{3}(2)^3 + 4(2)^2\right) \\
&= \frac{16}{3} - \frac{128}{3} + 64 + \frac{16}{3} - 16 \\
&= 16
\end{aligned}$$

(b) Use a change of variables to rewrite R as a square in the uv-plane and evaluate.

Solution: If we rewrite the edges as $x - y = 0$, $x - y = 4$, $x + y = 0$ and $x + y = 4$ and assign $u = x - y$ and $v = x + y$ then the new region in the uv-plane has edges $u = 0$, $u = 4$, $v = 0$ and $v = 4$.

We can solve for x and y by adding to get $u + v = 2x$ and so $x = \frac{1}{2}u + \frac{1}{2}v$ and then $y = v - x = -\frac{1}{2}u + \frac{1}{2}v$. The Jacobian is then:

$$J = \begin{vmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{vmatrix} = 1/2$$

Then the integral is:

$$\begin{aligned}
\iint_R x\, dA &= \iint_S \frac{1}{2}u + \frac{1}{2}v\, dA \\
&= \int_0^4 \int_0^4 \left(\frac{1}{2}u + \frac{1}{2}v\right)\left(\frac{1}{2}\right) dv\, du \\
&= \int_0^4 \frac{1}{4}uv + \frac{1}{8}v^2\Big|_0^4 du \\
&= \int_0^4 u + 2\, du \\
&= \frac{1}{2}u^2 + 2u\Big|_0^4 \\
&= 16
\end{aligned}$$

(c) These values should be the same. Are they?

Solution: Yep!