1. Consider the integrals: \[ A = \int_0^1 \int_0^x x^2 y \, dy \, dx \quad \text{and} \quad B = \int_0^2 \int_0^x x^2 y \, dy \, dx \]

(a) Either \( A < B \), \( A = B \) or \( A > B \). Without calculating either integral explain in a few sentences which is true and why. You may use pictures too if you feel it helps.

**Solution:** Since the function is positive we are finding the volume under it and since \( R \) (the base) for the integral \( B \) is larger than for the integral \( A \) we know that the volume for \( B \) is larger. Thus \( B > A \).

(b) Calculate both integrals. Were you correct?

**Solution:**

We have:

\[
A = \int_0^1 \int_0^x x^2 y \, dy \, dx = \int_0^1 \left[ \frac{1}{2} x^2 y^2 \right]_0^x \, dx = \int_0^1 \frac{1}{2} x^4 \, dx = \frac{1}{10} x^5 \bigg|_0^1 = \frac{1}{10}
\]

\[
B = \int_0^2 \int_0^x x^2 y \, dy \, dx = \int_0^2 \left[ \frac{1}{2} x^2 y^2 \right]_0^x \, dx = \int_0^2 \frac{1}{2} x^4 \, dx = \frac{1}{10} x^5 \bigg|_0^2 = \frac{32}{10}
\]

So yes, I was right!
2. Let $R$ be the region inside $r = 2 \cos \theta$ and to the right of $x = 1$. Consider the integral: \[ \iint_{R} x \, dA \]

(a) Draw a picture of $R$.
   \textbf{Solution:} Omitted here but it’s the right half of a circle of radius 1 centered at $(1, 0)$.

(b) Parametrize $R$ as polar and write down the corresponding iterated integral. Do not evaluate.
   \textbf{Solution:} We have:
   \[ \iiint_{R} x \, dA = \int_{\pi/4}^{\pi/4} \int_{-\pi/4}^{\pi/4} \int_{\sec \theta}^{2 \cos \theta} (r \cos \theta) r \, dr \, d\theta \]

(c) Parametrize $R$ as vertically simple and write down the corresponding iterated integral. Do not evaluate.
   \textbf{Solution:} We have:
   \[ \iiint_{R} x \, dA = \int_{1}^{2} \int_{-\sqrt{1-(x-1)^2}}^{\sqrt{1-(x-1)^2}} x \, dy \, dx \]

(d) Parametrize $R$ as horizontally simple and write down the corresponding iterated integral. Do not evaluate.
   \textbf{Solution:} We have:
   \[ \iiint_{R} x \, dA = \int_{-1}^{1} \int_{1}^{1+\sqrt{1-y^2}} x \, dy \, dx \]
3. Let $D$ be the solid above the cone $\phi = \phi_0$ and inside the sphere of radius $\rho = \rho_0$. Here both $\phi_0$ and $\rho_0$ are unknown constants.

(a) Use a triple integral in spherical coordinates to find a formula for the volume of $D$. Your answer will have $\phi_0$ and $\rho_0$ in it.

**Solution:** The volume is:

\[
\iiint_D 1 \, dV = \int_0^{2\pi} \int_0^{\phi_0} \int_0^{\rho_0} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta
\]

\[
= \int_0^{2\pi} \int_0^{\phi_0} \frac{1}{3} \rho^3 \sin \phi \bigg|_0^{\rho_0} \, d\phi \, d\theta
\]

\[
= \int_0^{2\pi} \int_0^{\phi_0} \frac{1}{3} \rho_0^3 \sin \phi \, d\phi \, d\theta
\]

\[
= \int_0^{2\pi} -\frac{1}{3} \rho_0^3 \cos \phi \bigg|_0^{\phi_0} \, d\theta
\]

\[
= \int_0^{2\pi} \frac{1}{3} \rho_0^3 (-\cos \phi_0 + 1) \, d\theta
\]

\[
= \frac{1}{3} \rho_0^3 (1 - \cos \phi_0) \theta \bigg|_0^{2\pi}
\]

\[
= \frac{2\pi}{3} \rho_0^3 (1 - \cos \phi_0)
\]

(b) When $\phi_0 = \pi/2$ and $\rho_0 = 3$ what does $D$ look like? When you plug these into your formula do you get the answer you expect? Explain in a sentence or two.

**Solution:** In this case $D$ is a hemisphere of radius 3 so we would expect to get:

\[
\frac{1}{2} \left[ \frac{4}{3} \pi (3)^3 \right] = 18\pi
\]

If we plug the values into the formula we get:

\[
\frac{2\pi}{3} (3)^3 (1 - 0) = 18\pi
\]

So they match.

(c) When $\phi_0 = \pi$ and $\rho_0 = 5$ what does $D$ look like? When you plug these into your formula do you get the answer you expect? Explain in a sentence or two.

**Solution:** In this case $D$ is a sphere of radius 5 so we would expect to get:

\[
\frac{4}{3} \pi (5)^3 = \frac{500\pi}{3}
\]

If we plug the values into the formula we get:

\[
\frac{2\pi}{3} (5)^3 (1 - (-1)) = \frac{500\pi}{3}
\]

So they match.

(d) Explain in a few sentences why cylindrical coordinates would be a really difficult way to do part (a).

**Solution:** There are several different ways this could be explained and anything reasonable is fine. One thing is that in cylindrical coordinates we’d need the radius of $R$, which arises from where the sphere meets the cone, and this would take some work. Moreover when $\phi_0 > \pi$ the sense of “top” and “bottom” functions is much more confusing and several separate integrals would be required.
4. Let \( R \) be the region bounded by the lines \( y = x, y = x - 4, y = -x \) and \( y = -x + 4 \). Consider the integral:

\[
\iint_{R} x \, dA
\]

(a) Parametrize \( R \) using two vertically simple regions and evaluate.

**Solution:** The top and bottom functions both change at \( x = 2 \) and so we need to split up the integral there. The result is:

\[
\iint_{R} x \, dA = \int_{0}^{2} \int_{-x}^{x} x \, dy \, dx + \int_{2}^{4} \int_{x-4}^{4-x} x \, dy \, dx 
\]

\[
= \int_{0}^{2} xy \bigg|_{-x}^{x} \, dx + \int_{2}^{4} xy \bigg|_{x-4}^{4-x} \, dx 
\]

\[
= \int_{0}^{2} 2x^2 \, dx + \int_{2}^{4} -2x^2 + 8x \, dx 
\]

\[
= \frac{2}{3} x^3 \bigg|_{0}^{2} + \left( -\frac{2}{3} x^3 + 4x^2 \right) \bigg|_{2}^{4} 
\]

\[
= \frac{2}{3} (2)^3 + \left( -\frac{2}{3} (4)^3 + 4(4)^2 \right) - \left( -\frac{2}{3} (2)^3 + 4(2)^2 \right) 
\]

\[
= \frac{16}{3} - \frac{128}{3} + 64 + \frac{16}{3} - 16 
\]

\[
= 16 
\]

(b) Use a change of variables to rewrite \( R \) as a square in the \( uv \)-plane and evaluate.

**Solution:** If we rewrite the edges as \( x - y = 0, x - y = 4, x + y = 0 \) and \( x + y = 4 \) and assign \( u = x - y \) and \( v = x + y \) then the new region in the \( uv \)-plane has edges \( u = 0, u = 4, v = 0 \) and \( v = 4 \).

We can solve for \( x \) and \( y \) by adding to get \( u + v = 2x \) and so \( x = \frac{1}{2} u + \frac{1}{2} v \) and then \( y = v - x = -\frac{1}{2} u + \frac{1}{2} v \). The Jacobian is then:

\[
J = \begin{vmatrix}
\frac{1}{2} & \frac{1}{2} \\
-\frac{1}{2} & \frac{1}{2}
\end{vmatrix} = 1/2 
\]

Then the integral is:

\[
\iint_{R} x \, dA = \int_{S} \frac{1}{2} u + \frac{1}{2} v \, dA 
\]

\[
= \int_{0}^{4} \int_{0}^{4} \left( \frac{1}{2} u + \frac{1}{2} v \right) \left( \frac{1}{2} \right) \, dv \, du 
\]

\[
= \int_{0}^{4} \int_{0}^{4} \frac{1}{4} uv + \frac{1}{8} v^2 \, dv \, du 
\]

\[
= \int_{0}^{4} \left( \frac{1}{4} u^2 + \frac{1}{8} v^2 \right) \bigg|_{0}^{4} \, du 
\]

\[
= \int_{0}^{4} \left( \frac{1}{4} u^2 + 2u \right) \, du 
\]

\[
= \frac{1}{2} u^2 + 2u \bigg|_{0}^{4} 
\]

\[
= 16 
\]

(c) These values should be the same. Are they?

**Solution:** Yep!