MATH 241 Sections 03** Exam 3 Spring 2021

Exam Submission:

1. Submit this exam to Gradescope.
2. Tag your problems!
3. You may print the exam, write on it, scan and upload.
4. Or you may just write on it on a tablet and upload.
5. Or you are welcome to write the answers on a separate piece of paper if other options don't appeal to you, then scan and upload.

Exam Rules:

1. You may ask me for clarification on questions but you may not ask me for help on questions!
2. You are permitted to use any non-interactive resources. This includes books, static pages on the internet, your notes, and YouTube videos.
3. You are not permitted to use any interactive resources. This includes your friends, your friends' friends, your calculator, Matlab, Wolfram Alpha, and online chat groups.
   Exception: Calculators are fine for basic arithmetic.
4. If you are unsure about whether a resource is considered "interactive" simply ask me and I'll let you (and everyone) know.
5. Petting small animals for stress relief is acceptable and is not considered an "interactive resource".

Work Shown:

1. Show all work as appropriate for and using techniques learned in this course.
2. Any pictures, work and scribbles which are legible and relevant will be considered for partial credit.

1. (a) Write down a parameterization of the part of the plane $2x + y = 10$ in the first octant and below the plane $z = 7$. No sketch is required. [5 pts]

Solution:

The most natural parameterization seems to be:

$$\bar{\boldsymbol{r}}(x, z) = x\,\hat{\imath} + (10 - 2x)\,\hat{\jmath} + z\,\hat{k}$$

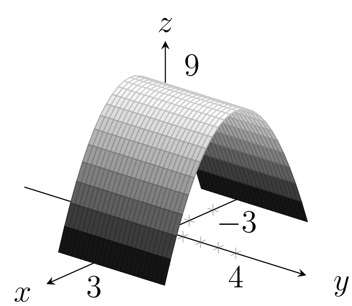
$$0 \le x \le 5$$

$$0 \le z \le 7$$

(b) Sketch the surface parameterized by: [5 pts]

$$\bar{\boldsymbol{r}}(x, y) = x\,\hat{\imath} + y\,\hat{\jmath} + (9 - x^2)\,\hat{k}$$
$$-2 \le y \le 4$$
$$-3 \le x \le 3$$

Solution:



2. The following integral can be evaluated without integrating but by understanding something about the shape and integrand. Describe the shape and give the numerical value for the integral. [10 pts]

$$\int_0^{2\pi} \int_0^2 \int_0^{5-r} r \, dz \, dr \, d\theta$$

Solution:

This is

$$\iiint_D 1 \, dV$$

in cylindrical where D is a cone of radius 2 and height 2 on top of a cylinder of radius 2 and height 3. The integral is the volume hence equals:

$$\frac{1}{3}\pi(2)^2(2) + \pi(2)^2(3)$$

3. Let D be the solid object inside the cylinder $(x-1)^2+y^2 = 1$, outside the cylinder $x^2 + y^2 = 1$, above the xy-plane and below the parabolic sheet $z = 9 - x^2$. If the density at a point is given by the function $f(x, y, z) = x$, set up but do not evaluate the iterated triple integral in cylindrical coordinates for the mass of D. [10 pts]

You Should Not Evaluate Your Resulting Integral!

Solution:

The cylinders are better in polar as $r = 1$ and $r = 2\cos\theta$ respectively. Note that they meet when $\cos\theta = \frac{1}{2}$ so $\theta = \pm\frac{\pi}{3}$.

The final result is:

$$\iiint_D x\, dV = \int_{-\pi/3}^{\pi/3} \int_1^{2\cos\theta} \int_0^{9-r^2\cos^2\theta} r\cos\theta\, r\, dz\, dr\, d\theta$$

4. Let D be the solid object inside both cones [10 pts]

$$z = \sqrt{3x^2 + 3y^2} \text{ and } z = 9 - \sqrt{x^2 + y^2}$$

and above the plane $z = 1$. Write down an iterated triple integral in spherical coordinates which yields the volume of D.

$\boxed{\textbf{You Should Not Evaluate Your Resulting Integral!}}$

Solution:

The θ values range from 0 to 2π.

The ϕ values range from 0 to $\pi/6$ as dictated by the first cone.

For ρ, the near (to the origin) function is the plane $z = 1$ which converts to $\rho\cos\theta = 1$ or $\rho = \sec\theta$. The far (from the origin) function is the cone $z = 9 - \sqrt{x^2 + y^2}$ which converts to $\rho\cos\theta = 9 - \rho\sin\theta$ or $\rho(\cos\theta + \sin\theta) = 9$ or $\rho = 9/(\cos\theta + \sin\theta)$.

The final result is:

$$\iiint_D 1\, dV = \int_0^{2\pi} \int_0^{\pi/6} \int_{\sec\theta}^{9/(\cos\theta+\sin\theta)} 1\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

5. The following iterated integral is impossible to evaluate as given: [15 pts]

$$\int_0^1 \int_0^{\sqrt{1-x^2}} e^{\left(1+2x^2+2y^2\right)}\, dy\, dx$$

Draw the corresponding region R and re-iterate the integral in a way that allows you to evaluate it.

$\boxed{\textbf{You Should Evaluate Your Resulting Integral!}}$

Solution:

The region here is the quarter-disk of radius 1 in the first quadrant. If we parametrize using polar instead we get:

$$\begin{aligned}
\iint_D e^{\left(1+2x^2+2y^2\right)}\, dA &= \int_0^{\pi/2} \int_0^1 e^{1+2r^2} r\, dr\, d\theta \\
&= \int_0^{\pi/2} \frac{1}{4} e^{1+2r^2}\Big|_0^1\, d\theta \\
&= \int_0^{\pi/2} \frac{1}{4} e^3 - \frac{1}{4} e^1\, d\theta \\
&= \frac{\pi}{8}(e^3 - e^1)
\end{aligned}$$

6. **Instruction:**

Let A be the sum of the digits of your UID.
Let B be the largest single digit of your UID.

Write down your UID and the value(s) and mark them clearly. In the problem below, replace them by the appropriate value(s) before proceeding.

Let R be the filled-in triangle with corners $(0,0)$, (A,B) and $(2A,0)$. [15 pts]

Set up and evaluate the double integral:

$$\iint_R 2Bx\,dA$$

You Should Evaluate and Simplify Your Resulting Integral!

Solution:

The three lines bounding the region are:

$$y=0,\; y=\tfrac{B}{A}x,\; y=2B-\tfrac{B}{A}x$$

If we parametrize as horizontally simple then the left and right functions are:

Left: $x=\frac{A}{B}y$ Right: $x=2A-\frac{A}{B}y$

The y-values go from 0 to B and so the integral is:

$$\begin{aligned}
\iint_R \frac{1}{2}Bx\,dA &= \int_0^B \int_{\frac{A}{B}x}^{2A-\frac{A}{B}x} 2Bx\,dx\,dy \\
&= \int_0^B Bx^2\Big|_{\frac{A}{B}x}^{2A-\frac{A}{B}x} dy \\
&= B\int_0^B \left(\frac{A}{B}y\right)^2 - \left(2A-\frac{A}{B}y\right)^2 dy \\
&= B\int_0^B 4A^2 + 2\frac{A^2}{B}y\,dy \\
&= 4A^2By + \frac{A^2}{B}y^2\Big|_0^B \\
&= 4A^2B(B) + \frac{A^2}{B}B^2 \\
&= 4A^2B^2 + A^2B
\end{aligned}$$

7. **Instruction:**

Let B be the sum of the two smallest nonzero digits of your UID.

Write down your UID and the value(s) and mark them clearly. In the problem below, replace them by the appropriate value(s) before proceeding.

Suppose R is the region bounded by the three lines $y = x + 1$, $y = Bx$ and $y = (B + 1)x$. [15 pts]

Apply the change of variables $x = u + v$ and $y = u + (B + 1)v$ to the following integral:

$$\iint_R x\, dR$$

Rewrite the resulting integral as an iterated double integral in the uv-plane.

$$\boxed{\textbf{You Should Not Evaluate Your Resulting Integral!}}$$

Solution:

The three lines change as follows:

$y = x + 1 \Rightarrow u + (B + 1)v = u + v + 1 \Rightarrow Bv = 1 \Rightarrow v = 1/B$

$y = Bx \Rightarrow u + (B + 1)v = B(u + v) \Rightarrow v = (B - 1)u$

$y = (B + 1)x \Rightarrow u + (B + 1)v = (B + 1)(u + v) \Rightarrow u = 0$

Thus the new region S is bounded by these lines. Note that the lines $v = 1/B$ and $v = (B - 1)u$ meet when $(B - 1)u = 1/B$ so $u = 1/(B^2 - B)$.

The Jacobian is:

$$\begin{vmatrix} 1 & 1 \\ 1 & B+1 \end{vmatrix} = (B + 1) - 1 = B$$

Hence we have:

$$\begin{aligned} \iint_R x\, dR &= \iint_S (u + v)|B|\, dA \\ &= B \int_0^{1/(B^2-B)} \int_{(B-1)u}^{1/B} u + v\, dv\, du \end{aligned}$$

8. **Instruction:**

Let C be the sum of the first four (leftmost) digits of your UID.

Write down your UID and the value(s) and mark them clearly. In the problem below, replace them by the appropriate value(s) before proceeding.

Let D be the solid object bounded by the four surfaces: [15 pts]

$$\begin{aligned} x &= C^2 - y^2 \\ x &= y + C \\ z &= x \\ z &= 2x \end{aligned}$$

If the density at a point is given by the function $f(x, y, z) = z$, set up but do not evaluate the iterated triple integral in rectangular coordinates for the mass of D.

You Should Not Evaluate Your Resulting Integral!

Solution:

The sides of the region R are bounded by $x = C^2 - y^2$ and $x = y + C$ which meet at:

$$\begin{aligned} C^2 - y^2 &= y + C \\ y^2 + y + (C - C^2) &= 0 \\ y^2 + y - C(C-1) &= 0 \\ (y + C)(y - (C-1)) &= 0 \end{aligned}$$

Hence at $y = -C$ and $y = C - 1$.

The mass is then:

$$\text{Mass} = \iiint_D z\, dV = \int_{-C}^{C-1} \int_{y+C}^{C^2-y^2} \int_x^{2x} z\, dz\, dx\, dy$$