MATH241 Spring 2023 Exam 3 (Justin W-G) Solutions

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Instructions:

1. Please do all problems on the pages and in the spaces provided. This exam will be scanned into Gradescope and if your answers are not in the correct locations they will not be found or graded!
2. Only simplify Calculus 3 related calculations.

1. Write T for True or F for False in the box to the right. No justification is required. Unreadable or ambiguous letters will be marked as incorrect. [10 pts]

Solution:

Statement	T or F
Every region in the xy plane is vertically simple.	F
The integral $\iiint_D f(x,y,z)\,dV$ can represent mass.	T
Spherical coordinates can be used to determine volume.	T
$\int_0^1 \int_1^2 f(x,y)\,dy\,dx = \int_1^2 \int_0^1 f(x,y)\,dx\,dy$	T
The Jacobian which appears in spherical coordinates is $\rho \sin^2 \phi$.	F

2. The following integrals can be evaluated without integrating but by understanding something about the shapes and integrand. Briefly describe each shape and give the value of the integral.

NOTE: Zero points will be given for actually integrating!

(a) This integral: [5 pts]

$$\int_0^1 \int_0^y 2\,dx\,dy$$

Solution:
This is the volume of a solid whose base is a triangle and has height 2, so it is:

$$\frac{1}{2}(1)(1)2 = 1$$

(b) This integral: [5 pts]

$$\int_0^4 \int_0^{4-x} \int_0^3 7\,dz\,dy\,dx$$

Solution:
This is the mass of a solid whose base is a triangle and has height 7, so it is:

$$7\left(\frac{1}{2}\right)(4)(4)(3)$$

3. Evaluate this integral: [5 pts]

$$\int_0^1 \int_1^{2-x} x\, dy\, dx$$

Solution:

We have:

$$\begin{aligned}
\int_0^1 \int_1^{2-x} x\, dy\, dx &= \int_0^1 xy\Big|_1^{2-x} dx \\
&= \int_0^1 x(2-x) - x(1)\, dx \\
&= \int_0^1 x - x^2\, dx \\
&= \frac{1}{2}x^2 - \frac{1}{3}x^3\Big|_0^1 \\
&= \frac{1}{6}
\end{aligned}$$

4. Suppose R is the region in the first quadrant between $y = \frac{1}{8}x^2$ and $y = \sqrt{x}$. Set up the iterated double integral in rectangular coordinates using a vertically simple region for: [10 pts]

$$\iint_R xy\, dA$$

DO NOT EVALUATE!

Solution:

The functions meet when:

$$\begin{aligned}
\frac{1}{8}x^2 &= \sqrt{x} \\
x^2 &= 8\sqrt{x} \\
x^4 &= 64x \\
x(x^3 - 64) &= 0
\end{aligned}$$

Thus $x = 0$ and $x = 4$, so the integral is:

$$\int_0^4 \int_{\frac{1}{8}x^2}^{\sqrt{x}} xy\, dy\, dx$$

5. Suppose R is the region inside $r = \cos\theta$ and outside $r = \sqrt{3}\sin\theta$ and in the first quadrant. Set up the iterated double integral in polar coordinates for: [10 pts]

$$\iint_R x - y\, dA$$

DO NOT EVALUATE!

Solution:

The two meet at:

$$\begin{aligned} \cos\theta &= \sqrt{3}\sin\theta \\ \tan\theta &= \frac{1}{\sqrt{3}} \\ \theta &= \frac{\pi}{6} \end{aligned}$$

Thus:

$$\int_0^{\pi/6} \int_{\sqrt{3}\sin\theta}^{\cos\theta} (r\cos\theta - r\sin\theta) r\, dr\, d\theta$$

6. Suppose D is the solid between $r = \sin\theta$ and $r = 2\sin\theta$ and between $z = 1$ and $z = 10$. Write down the iterated triple integral in cylindrical coordinates for the volume of D. [15 pts]

DO NOT EVALUATE!

Solution:

We have:

$$\int_0^{\pi} \int_{\sin\theta}^{2\sin\theta} \int_1^{10} r\, dz\, dr\, d\theta$$

7. Suppose D is the solid inside the cylinder $x^2 + y^2 = 9$ and between the cones $z = \sqrt{x^2 + y^2}$ and $z = -\sqrt{x^2 + y^2}$. If the density is given by $f(x, y, z) = z^2$, write down the iterated triple integral in spherical coordinates for the mass of D. [15 pts]

DO NOT EVALUATE!

Solution:

We have:

$$\int_0^{2\pi} \int_{\pi/4}^{3\pi/4} \int_0^{3\csc\phi} (\rho\cos\phi)^2 \rho^2 \sin\phi \, d\rho \, d\phi, \, d\theta$$

8. Write down a parameterization of the part of the plane $y = 4 - x$ in the first octant and between $z = 0$ and $z = 7$. No sketch is required. [5 pts]

Solution:

For example:

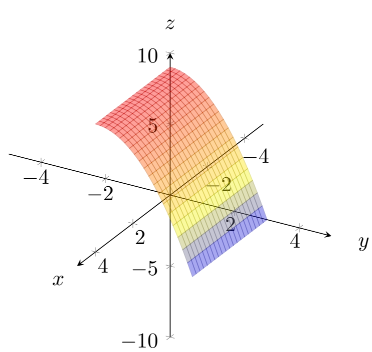
$$\begin{gathered} x\hat{\boldsymbol{\imath}} + (4-x)\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}} \\ 0 \le x \le 4 \\ 0 \le z \le 7 \end{gathered}$$

9. Sketch the surface parameterized by: [5 pts]

$$\begin{gathered} \bar{\boldsymbol{r}}(x,y) = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + (9-y^2)\hat{\boldsymbol{k}} \\ 0 \le x \le 4 \\ 0 \le y \le 3 \end{gathered}$$

Solution:

This is a part of the parabolic sheet in the first octant between $x = 0$ and $x = 4$.



10. Let R be the region bounded by the lines $y = 1$, $y = \frac{1}{4}x$, and $x - 3y = 2$. Consider the integral: [15 pts]

$$\iint_R \frac{y}{x - 3y}\, dA$$

Use the substitution $x = 3u + v$ and $y = u$ to convert this integral to an iterated integral in the uv-plane.

DO NOT EVALUATE!

Solution:

We convert the regions:

$y = 1 \implies u = 1$

$y = \frac{1}{4}x \implies u = \frac{1}{4}(3u + v) \implies v = u$

$x - 3y = 2 \implies 3u + v - 3u = 2 \implies v = 2$

The Jacobian is:

$$\begin{vmatrix} 3 & 1 \\ 1 & 0 \end{vmatrix} = -1$$

Then we have:

$$\iint_R \frac{y}{x - 3y}\, dA = \iint_S \frac{u}{3u + v - 3u} |-1|\, dA = \int_1^2 \int_v^2 \frac{u}{v}\, dA$$