Math 241 Exam 4 Fall 2018 Solutions **Justin Wyss-Gallifent**

1. (a) Let $f(x, y, z) = x^2 + yz^3$. Only one of $\nabla \times (\nabla f)$ and $\nabla \times (\nabla \cdot f)$ makes sense. Calculate the one that does. [10 pts]

   Solution:

   Only the first makes sense and the result gives 0 becuase ∇f is a conservative vector field.

   (b) Evalute $\int_C x + y\, ds$ where C is the part of the semicircle $x^2 + y^2 = 9$ above the x-axis. [10 pts]

   Solution:

   We parametrize C by $\mathbf{r}(t) = 3\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j}$ for $0 \leq t \leq \pi$.
   Then $\mathbf{r}'(t) = -3\sin t\,\mathbf{i} + 3\cos t\,\mathbf{j}$ and $||\mathbf{r}'(t)|| = 3$. Then:

$$\begin{aligned}
\int_C x + y\, ds &= \int_0^\pi (3\cos t + 3\sin t)3\, dt \\
&= 9\sin t - 9\cos t\Big|_0^\pi \\
&= (9(0) - 9(-1)) - (9(0) - 9(1)) \\
&= 18
\end{aligned}$$

2. (a) Evaluate $\int_C (2xy + y)\,dx + (x^2 + x)\,dy$ where C is parametrized by $\mathbf{r}(t) = t^2 e^t\,\mathbf{i} + \sqrt{t}\,\mathbf{j}$ for $0 \le t \le 4$. [10 pts]

Solution:

Since $\mathbf{F}(x, y) = (2xy + y)\,\mathbf{i} + (x^2 + x)\,\mathbf{j}$ is conservative with potential function

$$f(x, y) = x^2 y + xy$$

we can use FTOLI.

The curve starts at $\mathbf{r}(0) = 0\,\mathbf{i} + 0\,\mathbf{j}$ or $(0, 0)$ and ends at $\mathbf{r}(4) = 16e^4\,\mathbf{i} + 2\,\mathbf{j}$ or $(16e^4, 2)$.

Therefore

$$\int_C (2xy + y)\,dx + (x^2 + x)\,dy = f(16e^4, 2) - f(0, 0) = (16e^4)^2(2) + (16e^4)(2)$$

(b) Evaluate $\int_C (x\,\mathbf{i} + x\,\mathbf{j}) \cdot d\mathbf{r}$ where C is the line segment from $(0, 0)$ to $(4, 2)$. [10 pts]

Solution:

We parametrize the line segment by $\mathbf{r}(t) = 4t\,\mathbf{i} + 2t\,\mathbf{j}$ for $0 \le t \le 1$. Then $\mathbf{r}'(t) = 4\,\mathbf{i} + 2\,\mathbf{j}$ and so:

$$\begin{aligned}
\int_C (x\,\mathbf{i} + x\,\mathbf{j}) \cdot d\mathbf{r} &= \int_0^1 (4t\,\mathbf{i} + 4t\,\mathbf{j}) \cdot (4\,\mathbf{i} + 2\,\mathbf{j})\,dt \\
&= \int_0^1 24t\,dt \\
&= 12t^2 \Big|_0^1 \\
&= 12
\end{aligned}$$

3. Evaluate $\int_C 3y\,dx + x^2\,dy$ where C is the triangle with vertices $(0,0)$, $(0,4)$ and $(2,4)$, oriented clockwise. [20 pts]

Solution: By Green's Theorem and due to the orientation:

$$\int_C 3y\,dx + x^2\,dy = -\iint_R 2x - 3\,dA$$

where R is the filled-in triangular region. Then we parametrize and proceed:

$$\begin{aligned}
-\iint_R 2x - 3\,dA &= -\int_0^2 \int_{2x}^4 2x - 3\,dy\,dx \\
&= -\int_0^2 2xy - 3y\Big|_{2x}^{4}\,dx \\
&= -\int_0^2 (2x(4) - 3(4)) - (2x(2x) - 3(2x))\,dx \\
&= -\int_0^2 -4x^2 + 14x - 12\,dx \\
&= \int_0^2 4x^2 - 14x + 12\,dx \\
&= \frac{4}{3}x^3 - 7x^2 + 12x\Big|_0^2 \\
&= \frac{4}{3}(2)^3 - 7(2)^2 + 12(2)
\end{aligned}$$

4. Let Σ be the part of the cylinder $x^2 + y^2 = 4$ in the first octant and below $z = 4$. Let C be the edge of Σ with counterclockwise orientation when viewed looking toward the origin. Apply Stokes' Theorem to the integral $\int_C x^2 z\,dx + y\,dy + xy^2\,dz$ and proceed until you have an iterated double integral. **Do Not Evaluate.** [20 pts]

Solution:

First, the curl of the vector field $\mathbf{F}$ is:

$$\nabla \times \mathbf{F} = 2xy\,\mathbf{i} - (y^2 - x^2)\,\mathbf{j} + 0\,\mathbf{k}$$

Then by Stokes' Theorem:

$$\int_C x^2 z\,dx + y\,dy + xy^2\,dz = \iint_\Sigma \left(2xy\,\mathbf{i} - (y^2 - x^2)\,\mathbf{j} + 0\,\mathbf{k}\right) \cdot \mathbf{n}\,dS$$

where Σ is given with orientation forward and to the right.

We then parametrize Σ by

$$\mathbf{r}(\theta, z) = 2\cos\theta\,\mathbf{i} + 2\sin\theta\,\mathbf{j} + z\,\mathbf{k} \text{ with } 0 \le \theta \le \pi/2 \text{ and } 0 \le z \le 4$$

From here

$$\begin{aligned}
\mathbf{r}_\theta &= -2\sin\theta\,\mathbf{i} + 2\cos\theta\,\mathbf{j} + 0\,\mathbf{k} \\
\mathbf{r}_z &= 0\,\mathbf{i} + 0\,\mathbf{j} + 1\,\mathbf{k} \\
\mathbf{r}_\theta \times \mathbf{r}_z &= 2\cos\theta\,\mathbf{i} + 2\sin\theta\,\mathbf{j} + 0\,\mathbf{k}
\end{aligned}$$

Since $0 \le \theta \le \pi/2$ both the $\mathbf{i}$, $\mathbf{j}$-components are positive so this matches Σ's orientation. Thus

$\iint_\Sigma \left(2xy\,\mathbf{i} - (y^2 - x^2)\,\mathbf{j} + 0\,\mathbf{k}\right) \cdot \mathbf{n}\,dS$

$$\begin{aligned}
&= \iint_R \left(2(2\cos\theta)(2\sin\theta)\,\mathbf{i} - (4\sin^2\theta - 4\cos^2\theta)\,\mathbf{j} + 0\,\mathbf{k}\right) \cdot (2\cos\theta\,\mathbf{i} + 2\sin\theta\,\mathbf{j} + 0\,\mathbf{k})\,dA \\
&= \int_0^{\pi/2} \int_0^4 16\sin\theta\cos^2\theta - (4\sin^2\theta - 4\cos^2\theta)(2\sin\theta)\,dz\,d\theta
\end{aligned}$$

5. (a) Let Σ be the part of the paraboloid $z = x^2 + y^2$ constrained by $0 \le x \le 1$ and $0 \le y \le 2$. [10 pts]
Write down an iterated double integral for the surface area of Σ. **Do Not Evaluate.**

Solution: The surface area is given by $\iint_\Sigma 1\,dS$. We parametrize the surface with

$$\mathbf{r}(x,y) = x\,\mathbf{i} + y\,\mathbf{j} + (x^2+y^2)\,\mathbf{k} \text{ with } 0 \le x \le 1 \text{ and } 0 \le y \le 2$$

and then

$$\begin{aligned}
\mathbf{r}_x &= 1\,\mathbf{i} + 0\,\mathbf{j} + 2x\,\mathbf{k} \\
\mathbf{r}_y &= 0\,\mathbf{i} + 1\,\mathbf{j} + 2y\,\mathbf{k} \\
\mathbf{r}_x \times \mathbf{r}_y &= -2x\,\mathbf{i} - 2y\,\mathbf{j} + 1\,\mathbf{k} \\
||\mathbf{r}_x \times \mathbf{r}_y|| &= \sqrt{4x^2+4y^2+1}
\end{aligned}$$

And so

$$\begin{aligned}
\iint_\Sigma 1\,dS &= \iint_R \sqrt{4x^2+4y^2+1}\,dA \\
&= \int_0^1 \int_0^2 \sqrt{4x^2+4y^2+1}\,dy\,dx
\end{aligned}$$

(b) Apply the Divergence Theorem to $\iint_\Sigma \left(y\,\mathbf{i} + xy^2\,\mathbf{j} + 3z\,\mathbf{k}\right) \cdot \mathbf{n}\,dS$ where Σ is part of the cylinder $x^2+y^2=9$ between $z=0$ and $z=4$ with the disks which seal it off at the ends, oriented outwards. Proceed until you have an iterated integral. **Do Not Evaluate.** [10 pts]

Solution: By the Divergence Theorem if D is the solid cylinder then:

$$\begin{aligned}
\iint_\Sigma \left(y\,\mathbf{i} + xy^2\,\mathbf{j} + 3z\,\mathbf{k}\right) \cdot \mathbf{n}\,dS &= \iiint_D 0 + 2xy + 3\,dV \\
&= \int_0^{2\pi} \int_0^3 \int_0^4 [2(r\cos\theta)(r\sin\theta) + 3]\,r\,dz\,dr\,d\theta
\end{aligned}$$