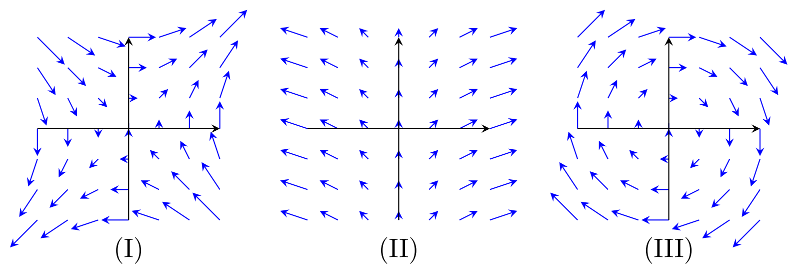
Math 241 Exam 4 Fall 2019 Solutions **Justin Wyss-Gallifent**

Please put problem 1 on answer sheet 1

1. Consider the following three pictures of vector fields. Each axis goes from -3 to 3 in each of the x and y directions and the vectors are anchored at integer coordinates. [20 pts]



(I) (II) (III)

For each of the following (a), (b), and (c), plug in a single point (x, y) which allows you to identify which of the vector fields (I), (II), or (III) matches the vector field.

(a) $\mathbf{F}(x, y) = 0.3y\,\mathbf{i} - 0.3x\,\mathbf{j}$

Solution:

For example $\mathbf{F}(3, 0) = 0\,\mathbf{i} - 0.9\,\mathbf{j}$ which matches (III).

(b) $\mathbf{F}(x, y) = 0.3y\,\mathbf{i} + 0.3x\,\mathbf{j}$

Solution:

For example $\mathbf{F}(3, 0) = 0\,\mathbf{i} + 0.9\,\mathbf{j}$ which matches (I).

(c) $\mathbf{F}(x, y) = 0.3x\,\mathbf{i} + 0.3\,\mathbf{j}$

Solution:

For example $\mathbf{F}(3, 0) = 0.9\,\mathbf{i} + 0.3\,\mathbf{j}$ which matches (II).

2. Parts (a) and (b) are unrelated.

(a) Let Σ be the sphere with equation $x^2 + y^2 + z^2 = 9$ with inwards orientation. Evaluate: [8 pts]

$$\iint_{\Sigma} (x\,\mathbf{i} + 2y\,\mathbf{j} + 3z\,\mathbf{k}) \cdot \mathbf{n}\,dS$$

Solution:
By the Divergence Theorem we have:

$$\begin{aligned}
\iint_{\Sigma} (x\,\mathbf{i} + 2y\,\mathbf{j} + 3z\,\mathbf{k}) \cdot \mathbf{n}\,dS &= -\iiint_{D} (1+2+3)\,dV \\
&= -6 \iiint_{D} 1\,dV \\
&= -6(\text{Volume}) \\
&= -6\left(\frac{4}{3}\pi(3)^2\right)
\end{aligned}$$

(b) Suppose C is the curve parametrized by $\mathbf{r}(t) = t^2\,\mathbf{i} + t\,\mathbf{j} + 2\,\mathbf{k}$ for $0 \le t \le 2$. Evaluate: [12 pts]

$$\int_C y\,dx + z\,dy + z\,dz$$

Solution:
We use the parametrization:

$$\begin{aligned}
\int_C y\,dx + z\,dy + z\,dz &= \int_0^2 (t)(2t) + (2)(1) + (2)(0)\,dt \\
&= \int_0^2 2t^2 + 2 \\
&= \frac{2}{3}t^3 + 2t\Big|_0^2 \\
&= \frac{2}{3}(2)^3 + 2(2)
\end{aligned}$$

3. Suppose Σ is the portion of the parabolic sheet $y = 4 - x^2$ in the first octant and below $z = 3$. [20 pts] The mass density at each point is given by $f(x, y, z) = x + y + z$. Set up an iterated integral for the mass of Σ.

Do Not Evaluate This Integral.

Solution:

We parametrize Σ:

$$\mathbf{r}(x, z) = x\,\mathbf{i} + (4 - x^2)\,\mathbf{j} + z\,\mathbf{k}$$
$$0 \le x \le 2$$
$$0 \le z \le 3$$

Then:

$$\begin{aligned}
\mathbf{r}_x &= 1\,\mathbf{i} - 2x\,\mathbf{j} + 0\,\mathbf{k} \\
\mathbf{r}_z &= 0\,\mathbf{i} + 0\,\mathbf{j} + 1\,\mathbf{k} \\
\mathbf{r}_x \times \mathbf{r}_z &= -2x\,\mathbf{i} - 1\,\mathbf{j} + 0\,\mathbf{k} \\
||\mathbf{r}_x \times \mathbf{r}_z|| &= \sqrt{4x^2 + 1}
\end{aligned}$$

Therefore:

$$\begin{aligned}
\text{Mass} &= \iint_\Sigma x + y + z\,dS \\
&= \iint_R \big(x + (4 - x^2) + z\big)\,\sqrt{4x^2 + 1}\,dA \\
&= \int_0^2 \int_0^3 \big(x + (4 - x^2) + z\big)\,\sqrt{4x^2 + 1}\,dz\,dx
\end{aligned}$$

4. Parts (a) and (b) are unrelated.

(a) Let C be the clockwise triangle with vertices $(5, 0)$, $(5, 10)$, and $(0, 10)$. Evaluate: [10 pts]

$$\int_C (x + 2y)\, dx + (4x + y)\, dy$$

Solution:
By Green's Theorem we have:

$$\int_C (x + 2y)\, dx + (4x + y)\, dy = -\iint_R 4 - 2\, dA$$

Where R is the filled-in triangle. This simplifies to:

$$-2 \iint_R 1\, dA = -2(\text{Area}) = -2\left(\frac{1}{2}(5)(10)\right)$$

(b) Let C be the straight-line segment from $(1, 2, -2)$ to $(8, 0, 5)$. Evaluate: [10 pts]

$$\int_C (y\,\mathbf{i} + x\,\mathbf{j} + 1\,\mathbf{k}) \cdot d\mathbf{r}$$

Solution:
The vector field is conservative with $f(x, y, z) = xy + z$. Thus we have:

$$\int_C (y\,\mathbf{i} + x\,\mathbf{j} + 1\,\mathbf{k}) \cdot d\mathbf{r} = f(8, 0, 5) - f(1, 2, -2) = [(8)(0) + 5] - [(1)(2) + (-2)]$$

5. Suppose C is the intersection of the plane $2x + y = 10$ with the cylinder $x^2 + z^2 = 9$ with counterclockwise orientation when viewed towards the origin from the positive y-axis. Apply Stokes' Theorem to the integral [20 pts]

$$\int_C (x+y)\,dx + y^2\,dy + (xyz)\,dz$$

and proceed until you have an iterated double integral.
Do Not Evaluate This Integral.

Solution:

By Stokes' Theorem we have:

$$\int_C (x+y)\,dx + y^2\,dy + (xyz)\,dz = \iint_\Sigma [xz\,\mathbf{i} - yz\,\mathbf{j} - 1\,\mathbf{k}] \cdot \mathbf{n}\,dS$$

Where Σ is the part of the plane inside the cylinder, oriented to the left and forward.

We parametrize Σ:

$$\mathbf{r}(\theta, r) = r\cos\theta\,\mathbf{i} + (10 - 2r\cos\theta)\,\mathbf{j} + r\sin\theta\,\mathbf{k}$$
$$0 \le \theta \le 2\pi$$
$$0 \le r \le 3$$

Then:

$$\begin{aligned}
\mathbf{r}_\theta &= -r\sin\theta\,\mathbf{i} + 2r\sin\theta\,\mathbf{j} + r\cos\theta\,\mathbf{j} \\
\mathbf{r}_r &= \cos\theta\,\mathbf{i} - 2\cos\theta\,\mathbf{j} + \sin\theta\,\mathbf{j} \\
\mathbf{r}_\theta \times \mathbf{r}_r &= 2r\,\mathbf{i} + r\,\mathbf{j} + 0\,\mathbf{k}
\end{aligned}$$

Which matches Σ's orientation. Thus we have:

$$\begin{aligned}
\iint_\Sigma [xz\,\mathbf{i} - yz\,\mathbf{j} - 1\,\mathbf{k}] \cdot \mathbf{n}\,dS &= \iint_R [(r\cos\theta)(r\sin\theta)\,\mathbf{i} - (10 - 2r\cos\theta)(r\sin\theta)\,\mathbf{j} - 1\,\mathbf{k}] \cdot (2r\,\mathbf{i} + r\,\mathbf{j} + 0\,\mathbf{k})\,dA \\
&= \int_0^{2\pi} \int_0^3 2r(r\sin\theta)(r\cos\theta) - r(10 - 2r\cos\theta)(r\sin\theta)\,dr\,d\theta
\end{aligned}$$