MATH241 Exam 4 Fall 2021 (Justin Wyss-Gallifent)

Solutions

1. Extra Credit: Write down your TA and the date, time, building code and room number of your final exam. [2 or 0 pts]

   Solution:

2. Suppose C is a piece of wire in the shape of a parabola with equation $y = 2x^2$ for $0 \leq x \leq 2$. Here x, y are in centimeters. If the electrical charge density at (x, y) is given by $f(x, y) = 4x$ coulombs per centimeter calculate the total electrical charge on the wire. [15 pts]

   Evaluate

   Solution:

   We have $\vec{r}(t) = t\hat{\boldsymbol{\imath}} + 2t^2\hat{\boldsymbol{\jmath}}$ for $0 \leq t \leq 2$. The total electrical charge is:

$$\begin{aligned}
\text{Charge} &= \int_C 4x\,ds \\
&= \int_0^2 4t||1\hat{\boldsymbol{\imath}} + 4t\hat{\boldsymbol{\jmath}}||\,dt \\
&= \int_0^2 4t\sqrt{1+16t^2}\,dt \\
&= \frac{1}{12}\left(1+16t^2\right)^{3/2}\Big|_0^2 \\
&= \frac{1}{12}(1+64)^{3/2} - \frac{1}{12}
\end{aligned}$$

3. Suppose an object follows the clockwise triangular path from $(0,0)$ to $(3,1)$ to $(1,0)$ and back to $(0,0)$. Use Green's Theorem to write down an iterated double integral for the work done on the object by the vector field: [15 pts]

$$\vec{F}(x,y) = 0.5y^2\hat{\boldsymbol{\imath}} - xy\hat{\boldsymbol{\jmath}}$$

Do Not Evaluate

Solution:

We are looking for $\int_C (0.5y^2\hat{\boldsymbol{\imath}} - xy\hat{\boldsymbol{\jmath}}) \cdot d\vec{r}$. By Green's Theorem we have:

$$\int_C (0.5y^2\hat{\boldsymbol{\imath}} - xy\hat{\boldsymbol{\jmath}}) \cdot d\vec{r} = -\iint_R -2y\,dA$$

If we parametrize R as a horizontally simple region then the left function is $x = 3y$ and the right function is $x = 2y+1$ and so the result is as follows.

$$\begin{aligned}
\int_C (0.5y^2\hat{\imath} - xy\hat{\boldsymbol{\jmath}}) \cdot \vec{r} &= -\iint_R -2y\,dA \\
&= \int_R 2y\,dA \\
&= \int_0^1 \int_{3y}^{2y+1} 2y\,dx\,dy
\end{aligned}$$

4. Suppose Σ is the portion of $y = 4 - x^2$ in the first octant and below $z = 5$. Write down an iterated double integral for the surface area of Σ. [15 pts]

Do Not Evaluate

Solution:

We parametrize Σ by:

$$\vec{r}(x,y) = x\hat{\boldsymbol{\imath}} + (4 - x^2)\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}$$
$$0 \leq x \leq 2$$
$$0 \leq y5$$

We then have:

$$\begin{aligned}
\vec{r}_x &= 1\hat{\boldsymbol{\imath}} - 2x\hat{\boldsymbol{\jmath}} + 0\hat{\boldsymbol{k}} \\
\vec{r}_z &= 0\hat{\boldsymbol{\imath}} + 0\hat{\boldsymbol{\jmath}} + 1\hat{\boldsymbol{k}} \\
\vec{r}_x \times \vec{r}_z &= -2x\hat{\boldsymbol{\imath}} - 1\hat{\boldsymbol{\jmath}} + 0\hat{\boldsymbol{k}} \\
||\vec{r}_x \times \vec{r}_z|| &= \sqrt{4x^2 + 1}
\end{aligned}$$

The surface area is then:

$$\begin{aligned}
\text{SA} &= \iint_{\Sigma} 1\, dS \\
&= \iint_{R} \sqrt{4x^2 + 1}\, dA \\
&= \int_0^2 \int_0^5 \sqrt{4x^2 + 1}\, dz\, dx
\end{aligned}$$

5. Let C be any curve from $(1,2,3)$ to $(4,3,-2)$. Evaluate and simplify: [10 pts]

$$\int_C \left(\frac{2x}{y} + ze^{xz}\right) dx + \left(-\frac{x^2}{y^2}\right) dy + xe^{xz}\, dz$$

Evaluate

Solution:

The vector field is conservative with potential function:

$$f(x,y,z) = \frac{x^2}{y} + e^{xz}$$

Thus by the Fundamental Theorem of Line Integrals we have:

$$\begin{aligned}\int_C \left(\frac{2x}{y} + ze^{xz}\right) dx + \left(-\frac{x^2}{y^2}\right) dy + xe^{xz}\, dz &= f(4,3,-2) - f(1,2,3) \\ &= \left[\frac{4^2}{3} + e^{(4)(-2)}\right] - \left[\frac{1^2}{2} + e^{(1)(3)}\right]\end{aligned}$$

6. Suppose Σ is the part of the cone $y = 3 - \sqrt{x^2+z^2}$ with $y \geq 0$ along with the base $x^2 + z^2 \leq 9$ on the xz-plane. Suppose Σ has inwards orientation. [10 pts]

Evaluate the integral:

$$\iint_\Sigma (4x\hat{\boldsymbol{\imath}} + 3y\hat{\boldsymbol{\jmath}} - z\hat{\boldsymbol{k}}) \cdot \vec{n}\, dS$$

Evaluate

Solution:

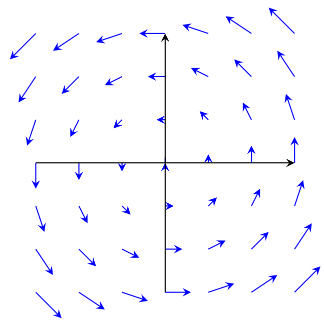
Since the surface is closed around the solid cone D we can use the Divergence Theorem with negation to get:

$$\iint_\Sigma (4x\hat{\boldsymbol{\imath}} + 3y\hat{\boldsymbol{\jmath}} - z\hat{\boldsymbol{k}}) \cdot \vec{n}\, dS = -\iiint_D 6\, dV$$

This is just -6 times the volume of the cone, hence:

$$-6\left(\frac{1}{3}\pi(3)^2(3)\right) = -54\pi$$

7. Explain briefly why the following vector field is not conservative: [5 pts]



Solution:

Imagine an object following a circular path around the origin and ending at the start point. The circular nature of the vector field indicates that work will be done, meaning the integral is not zero.

8. Suppose we have $f(x, y, z) = x^2y - xz$. Only one of the following makes sense. Circle the one that does and then calculate it. You do not need to justify how you made your choice. [10 pts]

$$\nabla \cdot (\nabla f) \qquad \text{and} \qquad \nabla(\nabla \cdot f)$$

Solution:

The one on the left makes sense and it equals:

$$\nabla \cdot (\nabla f) = \nabla \cdot ((2xy - z)\hat{\boldsymbol{\imath}} + x^2\hat{\boldsymbol{\jmath}} - x\hat{\boldsymbol{k}}) = 2y$$

9. Suppose C is the intersection curve of the paraboloid $z = x^2 + y^2$ with the cylinder $r = \cos\theta$ with clockwise orientation when viewed from above. Apply Stokes' Theorem to the integral [20 pts]

$$\int_C (x + z)\,dx + x^2\,dy + xy\,dz$$

and proceed until you have an iterated double integral.

Do Not Evaluate

Solution:

By Stokes' Theorem we have:

$$\int_C (x + z)\,dx + x^2\,dy + xy\,dz = \iint_\Sigma \left[x\hat{\boldsymbol{\imath}} - (y - 1)\hat{\boldsymbol{\jmath}} + 2x\hat{\boldsymbol{k}}\right] \cdot \vec{n}\,dS$$

Where Σ is the part of the paraboloid inside the cylinder, oriented downwards.

We parametrize Σ:

$$\vec{r}(\theta, r) = r\cos\theta\hat{\boldsymbol{\imath}} + r\sin\theta\hat{\boldsymbol{\jmath}} + r^2\hat{\boldsymbol{k}}$$
$$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$
$$0 \le r \le \cos\theta$$

Then:

$$\vec{r}_r = \cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\imath}} + 2r\hat{\boldsymbol{k}}$$
$$\vec{r}_\theta = -r\sin\theta\hat{\boldsymbol{\imath}} + r\cos\theta\hat{\boldsymbol{\jmath}} + 0\hat{\boldsymbol{k}}$$
$$\vec{r}_r \times \vec{r}_\theta = -2r^2\cos\theta\hat{\boldsymbol{\imath}} - 2r^2\sin\theta\hat{\boldsymbol{\jmath}} + r\hat{\boldsymbol{k}}$$

Which opposes Σ's orientation. Thus we have:

$\iint_\Sigma \left[x\hat{\boldsymbol{\imath}} - (y - 1)\hat{\boldsymbol{\jmath}} + 2x\hat{\boldsymbol{k}}\right] \cdot \vec{n}\,dS$

$$= -\iint_R \left[r\cos\theta\hat{\boldsymbol{\imath}} - (r\sin\theta - 1)\hat{\boldsymbol{\jmath}} + 2r\cos\theta\hat{\boldsymbol{k}}\right] \cdot \left[-2r^2\cos\theta\hat{\boldsymbol{\imath}} - 2r^2\sin\theta\hat{\boldsymbol{\jmath}} + r\hat{\boldsymbol{k}}\right]\,dA$$
$$= -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{\cos\theta} -2r^3\cos^2\theta + 2r^2\sin\theta(r\sin\theta - 1) + 2r^2\cos\theta\,dr\,d\theta$$