MATH241 Fall 2022 Exam 4 (Justin W-G) Solutions

NAME (Neatly):

UID (Neatly):

Instructions:

1. Please do all problems on the pages and in the spaces provided. This exam will be scanned into Gradescope and if your answers are not in the correct locations they will not be found or graded!
2. Only simplify Calculus 3 related calculations.

Extra Credit: Write the date, time, building and room of your final exam in the table below. If taking with ADS, write "ADS" in the Building and Room spaces. [≤ 2 pts]

Solution:

Date	Time	Building	Room

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(Except for doodling for stress relief.)

1. Write the word TRUE or FALSE in the box to the right. No justification is required. [10 pts]

Solution:

Statement	TRUE/FALSE
If C is closed and $\boldsymbol{F}$ is conservative then $\int_C \boldsymbol{F} \cdot d\boldsymbol{r} \neq 0$.	F
If $\boldsymbol{F}$ is conservative then $\nabla \times \boldsymbol{F} = \mathbf{0}$.	T
Green's Theorem applies if C is not closed.	F
We only have one way to evaluate $\int_C f\, ds$.	T
The expression $\nabla \times f(x, y, z)$ makes sense.	F

2. Let C be the circle in the xy-plane with radius 2 centered at $(8, 3)$. Let Σ be the part of the plane $z = 0$ with $0 \leq x \leq 5$ and $0 \leq y \leq 5$, oriented upwards. The following integrals can be evaluated without integrating. Write down the associated values to the right. No justification is required. [10 pts]

Solution:

Integral	Value
$\int_C 10\, ds$	40π
$\iint_\Sigma 10\, dS$	250
$\iint_\Sigma (4x\,\hat{\boldsymbol{\imath}} + 7xyz\,\hat{\boldsymbol{\jmath}}) \cdot \boldsymbol{n}\, dS$	0

3. The Fundamental Theorem of Line Integrals can be used to evaluate exactly one of the following. Use it for that one and use something else for the other one.

   (a) Suppose $\boldsymbol{F}(x,y,z) = z\,\hat{\boldsymbol{\imath}} + 2y\,\hat{\boldsymbol{\jmath}} + x\,\hat{\boldsymbol{k}}$ and C is the straight line segment from $(1,2,3)$ to $(6,8,10)$. Evaluate the line integral: [10 pts]

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$$

$$\boxed{\textbf{You Should Evaluate Your Result!}}$$

Solution:

The vector field is conservative with $f(x,y,z) = xz + y^2$ and so

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = z\,\hat{\boldsymbol{\imath}} + 2y\,\hat{\boldsymbol{\jmath}} + x\,\hat{\boldsymbol{k}} = f(6,8,10) - f(1,2,3) = 117$$

   (b) Suppose $\boldsymbol{F}(x,y,z) = x\,\hat{\boldsymbol{\imath}} + x\,\hat{\boldsymbol{\jmath}} + y\,\hat{\boldsymbol{k}}$ and C is the straight line segment from $(1,2,3)$ to $(6,8,10)$. Evaluate the line integral: [10 pts]

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$$

$$\boxed{\textbf{You Should Evaluate Your Result!}}$$

Solution:

We parameterize with $\boldsymbol{r}(t) = (1+5t)\hat{\boldsymbol{\imath}} + (2+6t)\hat{\boldsymbol{\jmath}} + (3+7t)\hat{\boldsymbol{k}}$ for $0 \le t \le 1$ so then $\boldsymbol{r}'(t) = 5\hat{\boldsymbol{\imath}} + 6\hat{\boldsymbol{\jmath}} + 7\hat{\boldsymbol{k}}$ and then:

$$\begin{aligned}
\int_C \boldsymbol{F} \cdot d\boldsymbol{r} &= \int_0^1 ((1+5t)\hat{\boldsymbol{\imath}} + (1+5t)\hat{\boldsymbol{\jmath}} + (2+6t)\hat{\boldsymbol{k}}) \cdot (5\hat{\boldsymbol{\imath}} + 6\hat{\boldsymbol{\jmath}} + 7\hat{\boldsymbol{k}})\,dt \\
&= \int_0^1 5 + 25t + 6 + 30t + 14 + 42t\,dt \\
&= \int_0^1 97t + 25\,dt \\
&= \frac{97}{2} + 25
\end{aligned}$$

4. Let Σ be the part of the plane $2x + y = 4$ in the first octant and below $z = 5$. If the mass density at a point is given by $f(x, y, z) = y$, write down and evaluate the integral for the mass of Σ. [15 pts]

You Should Evaluate Your Result!

Solution:

We parameterize with:

$$\begin{gathered}\boldsymbol{r}(x,z) = x\hat{\boldsymbol{\imath}} + (4-2x)\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}\\ 0 \le x \le 2\\ 0 \le y \le 5\end{gathered}$$

Then:

$$\begin{aligned}\boldsymbol{r}_x &= 1\hat{\boldsymbol{\imath}} - 2\hat{\boldsymbol{\jmath}} + 0\hat{\boldsymbol{k}}\\ \boldsymbol{r}_z &= 0\hat{\boldsymbol{\imath}} + 0\hat{\boldsymbol{\jmath}} + 1\hat{\boldsymbol{k}}\\ \boldsymbol{r}_x \times \boldsymbol{r}_y &= -2\hat{\boldsymbol{\imath}} - 1\hat{\boldsymbol{\jmath}} + 0\hat{\boldsymbol{k}}\\ ||\boldsymbol{r}_x \times \boldsymbol{r}_y|| &= \sqrt{5}\end{aligned}$$

Then:

$$\begin{aligned}\text{Mass} &= \iint_{\Sigma} y\,dS\\ &= \iint_{R} (4-2x)\sqrt{5}\,dA\\ &= \int_0^2\int_0^5 \sqrt{5}(4-2x)\,dz\,dx\\ &= \dots\\ &= 20\sqrt{5}\end{aligned}$$

5. Use Green's Theorem to evaluate the line integral $\int_C 4y\,dx + 6x\,dy$ where C is the curve traveling from $(0,0)$ to $(0,6)$ to $(12,0)$ and then back to $(0,0)$. [10 pts]

You Should Evaluate Your Result!

Solution:

By Green's Theorem with R being the filled in triangle and noting that C is clockwise:

$$\int_C 4y\,dx + 6x\,dy = -\iint_R 2\,dA = -2(\text{Area of } R) = -72$$

6. Let Σ be the part of $z = 2 - \sqrt{x^2 + y^2}$ above the xy-plane as well as the disk $x^2 + y^2 \leq 4$ in the xy-plane, oriented inwards. Apply the Divergence Theorem to construct an iterated triple integral in cylindrical coordinates for the following: [15 pts]

$$\iint_\Sigma (xy\,\hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}} + xz\,\hat{\boldsymbol{k}}) \cdot \boldsymbol{n}\, dS$$

$$\boxed{\textbf{You Should Not Evaluate Your Result!}}$$

Solution:

By the Divergence Theorem with D being the cond and noting that Σ is oriented inwards:

$$\begin{aligned}\iint_\Sigma (xy\,\hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}} + xz\,\hat{\boldsymbol{k}}) \cdot \boldsymbol{n}\, dS &= -\iiint_D y + 1 + z\, dV \\ &= -\int_0^{2\pi} \int_0^2 \int_0^{2-r} (r\sin\theta + 1 + r\cos\theta) r\, dz\, dr\, d\theta\end{aligned}$$

7. Let Σ be the part of the vertical plane $x + 2y = 4$ in the first octant and below $z = 5$. Let C be the edge of Σ with clockwise orientation when viewed from the first octant looking towards the origin. Consider the integral: [20 pts]

$$\int_C z^2\,dx + x\,dy + z\,dz$$

Use Stokes' Theorem to rewrite the line integral as a surface integral and then as an iterated double integral.

$$\boxed{\textbf{You Should Not Evaluate Your Result!}}$$

Solution:

Stokes' Theorem gives u:

$$\int_C z^2\,dx + x\,dy + z\,dz = \iint_\Sigma (0\hat{\boldsymbol{\imath}} + 2z\hat{\boldsymbol{\jmath}} + 1\hat{\boldsymbol{k}}) \cdot \boldsymbol{n}\,dS$$

Where Σ has the induced orientation away from where we are viewing,

We parameterize Σ using:

$$\begin{gathered}\boldsymbol{r}(y,z) = (4-2y)\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}\\ 0 \le y \le 2\\ 0 \le z \le 5\end{gathered}$$

Then we have:

$$\begin{aligned}\boldsymbol{r}_y &= -2\hat{\boldsymbol{\imath}} + 1\hat{\boldsymbol{\jmath}} + 0\hat{\boldsymbol{k}}\\ \boldsymbol{r}_z &= 0\hat{\boldsymbol{\imath}} + 0\hat{\boldsymbol{\jmath}} + 1\hat{\boldsymbol{k}}\\ \boldsymbol{r}_y \times \boldsymbol{r}_z &= 1\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}} + 0\hat{\boldsymbol{k}}\end{aligned}$$

Observing that this is the opposite of Σ's orientation we get:

$$\begin{aligned}\iint_\Sigma (0\hat{\boldsymbol{\imath}} + 2z\hat{\boldsymbol{\jmath}} + 1\hat{\boldsymbol{k}}) \cdot \boldsymbol{n}\,dS &= -\iint_R (0\hat{\boldsymbol{\imath}} + 2z\hat{\boldsymbol{\jmath}} + 1\hat{\boldsymbol{k}}) \cdot (1\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}} + 0\hat{\boldsymbol{k}})\,dA\\ &= -\int_0^2 \int_0^5 4z\,dz\,dy\end{aligned}$$