MATH241 Fall 2023 Exam 4 (Justin W-G) Solutions

Name (Neatly):	
UID (Neatly):	

Instructions:

1. Please do all problems on the pages and in the spaces provided. This exam will be scanned into Gradescope and if your answers are not in the correct locations they will not be found or graded!
2. Only simplify Calculus 3 related calculations unless otherwise specified.

1. Write TRUE or FALSE in the box to the right. No justification is required. Unreadable or ambiguous answers will be marked as incorrect. [10 pts]

Solution:

Statement	TRUE/FALSE
You absolutely must use FTOLI for integrating conservative vector fields.	FALSE
The surface integral of a vector field requires an oriented surface.	TRUE
Green's Theorem applies in 3D space.	FALSE
$\nabla \bar{\boldsymbol{F}}$ makes sense.	FALSE
Surface area is the surface integral of $1\,\hat{\boldsymbol{\imath}}$.	FALSE

2. Let C be the curve parametrized by $\bar{\boldsymbol{r}}(t) = 3\cos t\,\hat{\boldsymbol{\imath}} + 3\sin t\,\hat{\boldsymbol{\jmath}}$ for $0 \le t \le \frac{2\pi}{3}$. Without integrating, what is the value of $\int_C 7\,ds$ and why? [5 pts]

Solution:

The curve C is 1/3 of a circle of radius 3 and we are measuring the mass, assuming a density of 7. Since the length is:

$$\frac{1}{3}(2\pi 3) = 2\pi$$

The mass is then $7(2\pi) = 14\pi$.

3. Let Σ be the disk $x^2 + z^2 = 9$ in the plane $y = 42$, oriented in the positive y direction. Without integrating, what is the value of $\iint_\Sigma (10x\,\hat{\boldsymbol{\imath}} + 0\,\hat{\boldsymbol{\jmath}} + xy^2z\,\hat{\boldsymbol{k}}) \cdot n\,dS$ and why? [5 pts]

Solution:

The surface Σ is flat in the y sense and the fluid has no y component so it it not flowing through the surface. Thus the result is 0.

4. Let Σ be the part of the cylinder $y^2+z^2=9$ between $x=1$ and $x=3$. If the mass density at a point is given by $f(x,y,z)=x$, write down and evaluate the integral for the mass of Σ. [20 pts]

You Should Evaluate Your Result!

Solution:

If we parameterize Σ by:

$$\bar{\boldsymbol{r}}(x,z) = x\,\hat{\boldsymbol{\imath}} + 3\cos\theta\,\hat{\boldsymbol{\jmath}} + 3\sin\theta\,\hat{\boldsymbol{k}}$$
$$0 \le \theta \le 2\pi$$
$$1 \le x \le 3$$

Then we have:

$$\begin{aligned}
\bar{\boldsymbol{r}}_\theta &= 0\,\hat{\boldsymbol{\imath}} - 3\sin\theta\,\hat{\boldsymbol{\jmath}} + 3\cos\theta\,\hat{\boldsymbol{k}} \\
\bar{\boldsymbol{r}}_x &= 1\,\hat{\boldsymbol{\imath}} + 0\,\hat{\boldsymbol{\jmath}} + 0\,\hat{\boldsymbol{k}} \\
\bar{\boldsymbol{r}}_\theta \times \bar{\boldsymbol{r}}_x &= 0\,\hat{\boldsymbol{\imath}} + 3\cos\theta\,\hat{\boldsymbol{\jmath}} + 3\sin\theta\,\hat{\boldsymbol{k}} \\
||\bar{\boldsymbol{r}}_\theta \times \bar{\boldsymbol{r}}_x|| &= \sqrt{9\cos^2\theta + 9\sin^2\theta} = 3
\end{aligned}$$

Thus the result is:

$$\begin{aligned}
\int_C y\,dS &= \iint_R x(3)\,dS \\
&= \int_0^{2\pi}\int_1^3 3x\,dx\,d\theta \\
&= \int_0^{2\pi} \frac{3}{2}x^2\Big|_1^3\,d\theta \\
&= \int_0^{2\pi} \frac{3}{2}(3)^2 - \frac{3}{2}(1)^2\,d\theta \\
&= \int_0^{2\pi} 12\,d\theta \\
&= 24\pi
\end{aligned}$$

5. Suppose $\bar{\boldsymbol{F}}(x,y,z) = 1\,\hat{\boldsymbol{\imath}} + z\,\hat{\boldsymbol{\jmath}} + y\,\hat{\boldsymbol{k}}$ and C is the straight line segment from $(1,2,3)$ to $(8,10,100)$. [5 pts]
Evaluate the line integral:

$$\int_C \bar{\boldsymbol{F}} \cdot d\bar{\boldsymbol{r}}$$

$\boxed{\textbf{You Should Evaluate Your Result!}}$

Solution:

Observe that $\bar{\boldsymbol{F}}$ is conservative with potential function $f(x,y,z) = x + yz$. Thus by FTOLI we have:

$$\int_C \bar{\boldsymbol{F}} \cdot d\bar{\boldsymbol{r}} = f(8,10,100) - f(1,2,3) = (8+1000) - (1+6)$$

6. Use Green's Theorem to evaluate the line integral $\int_C 4y\,dx + 6x\,dy$ where C is the curve following $y = x^2$ from $(0,0)$ to $(2,4)$, then following a straight line to $(2,0)$ then following a straight line back to $(0,0)$. [15 pts]

$\boxed{\textbf{You Should Evaluate Your Result!}}$

Solution:

The curve C described is clockwise around the region R for which C is the edge and so by Green's Theorem:

$$\int_C 4y\,dx + 6x\,dy = -\iint_R 6 - 4\,dA$$

We parameterize R as vertically simple and then:

$$\begin{aligned}
-\iint_R 6 - 4\,dA &= -\int_0^2 \int_0^{x^2} 2\,dy\,dx \\
&= -\int_0^2 2y\Big|_0^{x^2}\,dx \\
&= -\int_0^2 2x^2\,dx \\
&= -\frac{2}{3}x^3\Big|_0^2 \\
&= -\frac{2}{3}(2)^3
\end{aligned}$$

7. Let Σ be the part of $z = 2 - \sqrt{x^2 + y^2}$ above the xy-plane as well as the disk $x^2 + y^2 \leq 4$ in the xy-plane, oriented inwards. Apply the Divergence Theorem to construct an iterated triple integral in cylindrical coordinates for the following: [10 pts]

$$\iint_\Sigma (xy\,\hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}} + xz\,\hat{\boldsymbol{k}}) \cdot \bar{\boldsymbol{n}}\,dS$$

$\boxed{\textbf{You Should Not Evaluate Your Result!}}$

Solution:

By the Divergence Theorem if D is the solid for which Σ is the surface then because of the inwards orientation we have:

$$\iint_\Sigma (xy\,\hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}} + xz\,\hat{\boldsymbol{k}}) \cdot \bar{\boldsymbol{n}}\,dS = -\iiint_D y + 1 + x\,dV = -\iiint_D x + y + 1\,dV$$

We parameterize D in cylindrical coordinates to get:

$$-\iiint_D x + y + 1\,dV = -\int_0^{2\pi}\int_0^2\int_0^{2-r} (r\cos\theta + r\sin\theta + 1)r\,dz\,dr\,d\theta$$

8. Let Σ be the part of the vertical plane $x + 2y = 4$ in the first octant and below $z = 5$. Let C be the edge of Σ with clockwise orientation when viewed from the first octant looking towards the origin. Consider the integral: [25 pts]

$$\int_C z^2\,dx + x\,dy + z\,dz$$

Use Stokes' Theorem to rewrite the line integral as a surface integral and then as an iterated double integral.

$$\boxed{\textbf{You Should Not Evaluate Your Result!}}$$

Solution:

We see that C is the edge of the surface Σ where Σ is given. By Stokes' Theorem we then have:

$$\begin{aligned}
\int_C z^2\,dx + x\,dy + z\,dz &= \iint_\Sigma ((0-0)\ \hat{\boldsymbol{\imath}} - (0-2z)\ \hat{\boldsymbol{\jmath}} + (1-0)\ \hat{\boldsymbol{k}}) \cdot \bar{\boldsymbol{n}}\,dS \\
&= \iint_\Sigma (0\ \hat{\boldsymbol{\imath}} + 2z\ \hat{\boldsymbol{\jmath}} + 1\ \hat{\boldsymbol{k}}) \cdot \bar{\boldsymbol{n}}\,dS
\end{aligned}$$

Here Σ has orientation out into the first octant, induced by C.

We then parameterize Σ by:

$$\begin{gathered}
\bar{\boldsymbol{r}}(y,z) = (4-2y)\ \hat{\boldsymbol{\imath}} + y\ \hat{\boldsymbol{\jmath}} + z\ \hat{\boldsymbol{k}} \\
0 \le y \le 2 \\
0 \le z \le 6
\end{gathered}$$

We have:

$$\begin{aligned}
\bar{\boldsymbol{r}}_y &= -2\,\hat{\boldsymbol{\imath}} + 1\,\hat{\boldsymbol{\jmath}} + 0\,\hat{\boldsymbol{k}} \\
\bar{\boldsymbol{r}}_z &= 0\,\hat{\boldsymbol{\imath}} + 0\,\hat{\boldsymbol{\jmath}} + 1\,\hat{\boldsymbol{k}} \\
\bar{\boldsymbol{r}}_y \times \bar{\boldsymbol{r}}_z &= 1\,\hat{\boldsymbol{\imath}} + 2\,\hat{\boldsymbol{\jmath}} + 0\,\hat{\boldsymbol{k}}
\end{aligned}$$

These do not match the orientation of Σ and hence, letting R represent the set of inequalities:

$$\begin{aligned}
\iint_\Sigma (0\,\hat{\boldsymbol{\imath}} + 2z\,\hat{\boldsymbol{\jmath}} + 1\,\hat{\boldsymbol{k}}) \cdot \bar{\boldsymbol{n}}\,dS &= -\iint_R (0\,\hat{\boldsymbol{\imath}} + 2z\,\hat{\boldsymbol{\jmath}} + 1\,\hat{\boldsymbol{k}}) \cdot (1\,\hat{\boldsymbol{\imath}} + 2\,\hat{\boldsymbol{\jmath}} + 0\,\hat{\boldsymbol{k}})\,dA \\
&= -\iint_R 4z\,dA \\
&= -\int_0^2 \int_0^5 4z\,dz\,dy
\end{aligned}$$