1. (a) Evalute $\int_C x\ ds$ where C is the line segment from $(1,1)$ to $(4,7)$. [10 pts]
This integral should be evaluated.
Solution: We parametrize C by $\mathbf{r}(t) = (1+3t)\,\mathbf{i} + (1+6t)\,\mathbf{j}$ for $0 \leq t \leq 1$. Then $\mathbf{r}'(t) = 3\,\mathbf{i} + 6\,\mathbf{j}$ and $||\mathbf{r}'(t)|| = \sqrt{9+36} = \sqrt{45}$ and so

$$\begin{aligned}\int_C x\ ds &= \int_0^1 (1+3t)\sqrt{45}\,dt \\ &= \sqrt{45}t + \frac{3}{2}t^2\Big|_0^1 \\ &= \sqrt{45}\left(1+\frac{3}{2}\right)\end{aligned}$$

(b) Let Σ be the sphere $x^2+y^2+z^2=9$, oriented inwards. Use the Divergence Theorem to evaluate $\iint_\Sigma (2x\,\mathbf{i} + 5z\,\mathbf{j} + 5z\,\mathbf{k}) \cdot \mathbf{n}\ dS$. [10 pts]
This integral should be evaluated.
Solution: By the DT we have:

$$\begin{aligned}\iint_\Sigma (2x\,\mathbf{i} + 5z\,\mathbf{j} + 5z\,\mathbf{k}) \cdot \mathbf{n}\ dS &= -\int\int\int_D 2+0+5\,dV \\ &= -7(\text{Volume of D}) \\ &= -7\left(\frac{4}{3}\pi(3)^3\right)\end{aligned}$$

2. Use Green's Theorem to evaluate $\int_C y^3\ dx - x^3\ dy$ where C is the edge of the semicircle $x^2 + y^2 \leq 9$ in the first quadrant, oriented counterclockwise. [20 pts]
This integral should be evaluated.

Solution: We have:

$$\begin{aligned}
\int_C y^3\ dx - x^3\ dy &= \int\int_R -3x^2 - 3y^2\, dA \\
&= \int_0^{\pi/2} \int_0^3 -3r^2(r)\, dr\, d\theta \\
&= \int_0^{\pi/2} -\frac{3}{4}r^4 \Big|_0^3\ d\theta \\
&= \int_0^{\pi/2} -\frac{3}{4}(3)^4\, d\theta \\
&= -\frac{3}{4}(81)\theta \Big|_0^{\pi/2} \\
&= -\frac{3}{4}(81)\left(\frac{\pi}{2}\right)
\end{aligned}$$

3. Evaluate $\int_C \left(2xy + \frac{1}{y}\right)\, dx + \left(x^2 - \frac{x}{y^2}\right)\, dy$ where C is the curve parametrized by $\mathbf{r}(t) = (t^2 + t)\,\mathbf{i} + (t^3 + 2)\,\mathbf{j}$ for $1 \le t \le 2$. [20 pts]

Solution: The vector field is conservative with potential function $f(x, y) = x^2 y + \frac{x}{y}$.

The start point is $\mathbf{r}(1) = 2\,\mathbf{i} + 3\,\mathbf{j}$ so $(2, 3)$.

The end point is $\mathbf{r}(2) = 6\,\mathbf{i} + 10\,\mathbf{j}$ so $(6, 10)$.

Thus

$$\begin{aligned}\int_C \left(2xy + \frac{1}{y}\right)\, dx + \left(x^2 - \frac{x}{y^2}\right)\, dy &= f(6, 10) - f(2, 3)\\ &= \left[(6)^2(10) + \frac{6}{10}\right] - \left[(2)^2(3) + \frac{2}{3}\right]\end{aligned}$$

4. Let C be the intersection of the cylinder $r = 2\sin\theta$ with the paraboloid $z = 9 - x^2 - y^2$ with counterclockwise orientation when viewed from above. Apply Stokes' Theorem to the line integral $\int_C x\ dx + xy\ dy + xz\ dz$. Parametrize the resulting surface and proceed until you have an iterated double integral. [20 pts]

Do Not Evaluate This Integral.

Solution: The curve C is the edge of Σ where Σ is the portion of the paraboloid inside the cylinder oriented up. Therefore by Stokes' Theorem:

$$\begin{aligned}\int_C x\ dx + xy\ dy + xz\ dz &= \int\int_\Sigma [(0-0)\,\mathbf{i} - (z-0)\,\mathbf{j} + (y-0)\,\mathbf{k}] \cdot \mathbf{n}\, dS \\ &= \int\int_\Sigma [0\,\mathbf{i} - z\,\mathbf{j} + y\,\mathbf{k}] \cdot \mathbf{n}\, dS\end{aligned}$$

We parametrize Σ by:

$$\mathbf{r}(r,\theta) = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j} + (9-r^2)\,\mathbf{k} \text{ for } 0 \le \theta \le \pi \text{ and } 0 \le r \le 2\sin\theta$$

Then

$$\begin{aligned}\mathbf{r}_r &= \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j} - 2r\,\mathbf{k} \\ \mathbf{r}_\theta &= -r\sin\theta\,\mathbf{i} + r\cos\theta\,\mathbf{j} + 0\,\mathbf{k} \\ \mathbf{r}_r \times \mathbf{r}_\theta &= 2r^2\cos\theta\,\mathbf{i} + 2r^2\sin\theta\,\mathbf{j} + r\,\mathbf{k}\end{aligned}$$

This matches Σ's orientation. Therefore:

$$\begin{aligned}\int\int_\Sigma [0\,\mathbf{i} - z\,\mathbf{j} + y\,\mathbf{k}] \cdot \mathbf{n}\, dS &= \int\int_R \left[0\,\mathbf{i} - (9-r^2)\,\mathbf{j} + (r\sin\theta)\,\mathbf{k}\right] \cdot \left[2r^2\cos\theta\,\mathbf{i} + 2r^2\sin\theta\,\mathbf{j} + r\,\mathbf{k}\right]\ dA \\ &= \int_0^\pi \int_0^{2\sin\theta} -(9-r^2)2r^2\sin\theta + (r\sin\theta)r\, dr\, d\theta\end{aligned}$$

5. Let Σ be the part of the plane $2x + y = 4$ in the first octant and between $z = 0$ and $z = 3$. [20 pts]
Parametrize the surface and write down the iterated integral corresponding to the surface integral $\iint_\Sigma xy\ dS$.
Do Not Evaluate This Integral.

Solution: We parametrize Σ by

$$\mathbf{r}(x,z) = x\,\mathbf{i} + (4-2x)\,\mathbf{j} + z\,\mathbf{k} \text{ for } 0 \le x \le 2 \text{ and } 0 \le z \le 3$$

Then

$$\begin{aligned}
\mathbf{r}_x &= 1\,\mathbf{i} - 2\,\mathbf{j} + 0\,\mathbf{k} \\
\mathbf{r}_z &= 0\,\mathbf{i} + 0\,\mathbf{j} + 1\,\mathbf{k} \\
\mathbf{r}_x \times \mathbf{r}_z &= -2\,\mathbf{i} - 1\,\mathbf{j} + 0\,\mathbf{k} \\
||\mathbf{r}_x \times \mathbf{r}_z|| &= \sqrt{5}
\end{aligned}$$

Therefore:

$$\begin{aligned}
\iint_\Sigma xy\ dS &= \int\int_R x(4-2x)\sqrt{5} \\
&= \int_0^2 \int_0^3 x(4-2x)\sqrt{5}\,dz\,dx
\end{aligned}$$