1. Parts (a) and (b) are unrelated.

(a) Is \( \mathbf{F}(x, y, z) = xy \mathbf{i} + y \mathbf{j} - 2xz \mathbf{k} \) conservative? Justify your answer. \([10 \text{ pts}]\)

Solution:

We find:

\[
\nabla \times \mathbf{F} = (0 - 0) \mathbf{i} - (-2z - 0) \mathbf{j} + (0 - x) \mathbf{k} \neq 0
\]

so \( \mathbf{F} \) is not conservative.

(b) Given \( f(x, y, z) = x^2z + \frac{z}{y} \) only one of the following makes sense. Calculate the one that \([10 \text{ pts}]\) does:

\[
\nabla (\nabla f) \quad \nabla \cdot (\nabla f) \quad \nabla \cdot (\nabla \times f)
\]

Solution: Only the middle one makes sense and we have:

\[
\nabla f = \left( 2xz + \frac{1}{y} \right) \mathbf{i} - \frac{x}{y^2} \mathbf{j} + x^2 \mathbf{k}
\]

\[
\nabla \cdot (\nabla f) = 2z + \frac{2x}{y^3}
\]
2. Let \( C \) be the curve with parametrization given by:

\[
\mathbf{r}(t) = t^2 \mathbf{i} + (1 - t) \mathbf{j} \text{ for } 1 \leq t \leq 3.
\]

The Fundamental Theorem of Line Integrals can be used for exactly one of the following. Use it for that one and then do the other one another way.

(a) Evaluate \( \int_{C} x \, dx + x \, dy \).  
\[ \text{Solution:} \]
We cannot use the FTOLI for this. Instead find:

\[
\mathbf{r}'(t) = 2t \mathbf{i} - \mathbf{j}
\]

and then:

\[
\int_{C} x \, dx + x \, dy = \int_{1}^{3} (t^2)(2t) + (t^2)(-1) \, dt
\]

\[
= \int_{1}^{3} 3t^3 - t^2 \, dt
\]

\[
= \left[ \frac{3}{4}t^4 - \frac{1}{3}t^3 \right]_{1}^{3}
\]

\[
= \left[ \frac{3}{4}(3)^4 - \frac{1}{3}(3)^3 \right] - \left[ \frac{3}{4} - \frac{1}{3} \right]
\]

(b) Evaluate \( \int_{C} y \, dx + x \, dy \).
\[ \text{Solution:} \]
The vector field is conservative with potential function

\[
f(x, y) = xy
\]

The start and end points of the curve are:

\[
\mathbf{r}(1) = 1 \mathbf{i} + 0 \mathbf{j} \text{ so (1, 0)}
\]

\[
\mathbf{r}(3) = 9 \mathbf{i} - 2 \mathbf{j} \text{ so (9, -2)}
\]

Thus:

\[
\int_{C} y \, dx + x \, dy = f(9, -2) - f(1, 0) = ...
\]
3. Parts (a) and (b) are unrelated.

(a) Use the Divergence Theorem to evaluate the following integral where $\Sigma$ is a cube with side length 3:

$$\int\int_{\Sigma} (2x \mathbf{i} + x \mathbf{j} + 10z \mathbf{k}) \cdot \mathbf{n} \, dS$$

**Solution:**

If $D$ is the solid cube contained within $\Sigma$ then:

$$\int\int_{\Sigma} (2x \mathbf{i} + x \mathbf{j} + 10z \mathbf{k}) \cdot \mathbf{n} \, dS = \int\int\int_D 2 + 0 + 10 \, dV = 12(\text{Volume of } D) = 12(3)^3$$

(b) Evaluate $\int_C x \, dy + x^2 \, dx$ where $C$ is triangle with corners (0, 0), (0, 4) and (2, 4), oriented clockwise.

**Solution:**

We use Green’s Theorem. Let $R$ be the region inside the triangle and then:

$$\int_C x \, dy + x^2 \, dx = -\int\int_R 2x - 0 \, dA$$

$$= -\int_0^2 \int_{2x}^4 2x \, dy \, dx$$

$$= -\int_0^2 2xy \bigg|_{2x}^4 \, dx$$

$$= -\int_0^2 2x(4) - 2x(2x) \, dx$$

$$= -\int_0^2 8x - 4x^2 \, dx$$

$$= \frac{4}{3}x^3 - 4x^2 \bigg|_0^2$$

$$= \frac{4}{3}(2)^3 - 4(2)^2$$
4. Let $\Sigma$ be the portion of the plane $x + z = 4$ in the first octant between $y = 3$ and $y = 5$. If the mass density at a point is $f(x, y, z) = yz$ write down an iterated integral corresponding to the mass of $\Sigma$. Do Not Evaluate This Integral.

Solution:
We parametrize the surface $\Sigma$ by:

$$
\mathbf{r}(x, y) = x \mathbf{i} + y \mathbf{j} + (4 - x) \mathbf{k}
$$

with

$$
0 \leq x \leq 4
$$

and

$$
3 \leq y \leq 5\right\} R
$$

We then have:

$$
\mathbf{r}_x = 1 \mathbf{i} + 0 \mathbf{j} - 1 \mathbf{k}
$$

$$
\mathbf{r}_y = 0 \mathbf{i} + 1 \mathbf{j} + 0 \mathbf{k}
$$

$$
\mathbf{r}_x \times \mathbf{r}_y = 1 \mathbf{i} + 0 \mathbf{j} + 1 \mathbf{k}
$$

Then the mass is:

$$
\iint_{\Sigma} yz \, dS = \iint_{R} \sqrt{2}y(4 - x) \, dA = \int_{0}^{4} \int_{3}^{5} \sqrt{2}y(4 - x) \, dy \, dx
$$
5. Let $\Sigma$ be the intersection of the cylinder $x^2 + z^2 = 4$ with the plane $x + z = 4$ with counter-clockwise orientation when viewed from the positive $y$-axis. Apply Stokes’ Theorem to the integral $\int_C xz\,dx + xy\,dy + y^2\,dz$ and proceed until you have an iterated double integral. **Do Not Evaluate This Integral.**

**Solution:**
We apply Stokes Theorem to get:

$$
\int_C xz\,dx + xy\,dy + y^2\,dz = \iint_\Sigma [(2y - 0)\,i - (0 - x)\,j + (1 - 0)\,k] \cdot n\,dS = \iint_\Sigma [2y\,i + x\,j + 1\,k] \cdot n\,dS
$$

Where $\Sigma$ is the portion of the plane inside the cylinder with orientation towards the positive $y$-axis.

We parametrize the surface $\Sigma$ by:

$$
\mathbf{r}(r, \theta) = r \cos \theta\,i + (4 - r \sin \theta)\,j + r \sin \theta\,k \\
0 \leq \theta \leq 2\pi \\
0 \leq r \leq 2 \\
\left\{ R \right\
$$

We then have:

$$
\mathbf{r}_r = \cos \theta\,i - \sin \theta\,j + r \sin \theta\,k \\
\mathbf{r}_\theta = -r \sin \theta\,i - r \cos \theta\,j + r \cos \theta\,k \\
\mathbf{r}_r \times \mathbf{r}_\theta = 0\,i - r\,j - r\,k
$$

This is opposite to $\Sigma$’s orientation so we introduce a minus. Thus:

$$
\iint_\Sigma [2y\,i + x\,j + 1\,k] \cdot n\,dS = -\iint_R [2(4 - r \sin \theta)\,i + (r \cos \theta)\,j + 1\,k] \cdot [0\,i - r\,j - r\,k]\,dA = -\int_0^{2\pi} \int_0^2 -r^2 \cos \theta - r\,dr\,d\theta
$$