Math 241 Exam 4 Spring 2019 Solutions **Justin Wyss-Gallifent**

1. Parts (a) and (b) are unrelated.

   (a) Is $\mathbf{F}(x, y, z) = xy\,\mathbf{i} + y\,\mathbf{j} - 2xz\,\mathbf{k}$ conservative? Justify your answer. [10 pts]

   Solution:

   We find:

   $$\nabla \times \mathbf{F} = (0-0)\,\mathbf{i} - (-2z-0)\,\mathbf{j} + (0-x)\,\mathbf{k} \neq \mathbf{0}$$

   so $\mathbf{F}$ is not conservative.

   (b) Given $f(x, y, z) = x^2z + \frac{x}{y}$ only one of the following makes sense. Calculate the one that does: [10 pts]

   $$\nabla(\nabla f) \qquad \nabla \cdot (\nabla f) \qquad \nabla \cdot (\nabla \times f)$$

   Solution: Only the middle one makes sense and we have:

   $$\nabla f = \left(2xz + \frac{1}{y}\right)\mathbf{i} - \frac{x}{y^2}\,\mathbf{j} + x^2\,\mathbf{k}$$
   $$\nabla \cdot (\nabla f) = 2z + \frac{2x}{y^3}$$

2. Let C be the curve with parametrization given by:

$$\mathbf{r}(t) = t^2\,\mathbf{i} + (1-t)\,\mathbf{j} \text{ for } 1 \le t \le 3.$$

The Fundamental Theorem of Line Integrals can be used for exactly one of the following. Use it for that one and then do the other one another way.

(a) Evaluate $\int_C x\,dx + x\,dy$. [10 pts]

Solution:

We cannot use the FTOLI for this. Instead find:

$$\mathbf{r}'(t) = 2t\,\mathbf{i} - 1\,\mathbf{j}$$

and then:

$$\begin{aligned}\int_C x\,dx + x\,dy &= \int_1^3 (t^2)(2t) + (t^2)(-1)\,dt \\ &= \int_1^3 3t^3 - t^2\,dt \\ &= \frac{3}{4}t^4 - \frac{1}{3}t^3 \Big|_1^3 \\ &= \left[\frac{3}{4}(3)^4 - \frac{1}{3}(3)^3\right] - \left[\frac{3}{4} - \frac{1}{3}\right]\end{aligned}$$

(b) Evaluate $\int_C y\,dx + x\,dy$. [10 pts]

Solution:

The vector field is conservative with potential function

$$f(x,y) = xy$$

The start and end points of the curve are:

$$\mathbf{r}(1) = 1\,\mathbf{i} + 0\,\mathbf{j} \text{ so } (1,0)$$
$$\mathbf{r}(3) = 9\,\mathbf{i} - 2\,\mathbf{j} \text{ so } (9,-2)$$

Thus:

$$\int_C y\,dx + x\,dy = f(9,-2) - f(1,0) = ...$$

3. Parts (a) and (b) are unrelated.

(a) Use the Divergence Theorem to evaluate the following integral where Σ is a cube with side length 3: [8 pts]

$$\iint_{\Sigma} (2x\,\mathbf{i} + x\,\mathbf{j} + 10z\,\mathbf{k}) \cdot \mathbf{n}\,dS$$

Solution:

If D is the solid cube contained within Σ then:

$$\iint_{\Sigma} (2x\,\mathbf{i} + x\,\mathbf{j} + 10z\,\mathbf{k}) \cdot \mathbf{n}\,dS = \iiint_D 2 + 0 + 10\,dV = 12(\text{Volume of} D) = 12(3)^3$$

(b) Evaluate $\int_C x\,dy + x^2\,dx$ where C is triangle with corners $(0,0)$, $(0,4)$ and $(2,4)$, oriented clockwise. [12 pts]

Solution:

We use Green's Theorem. Let R be the region inside the triangle and then:

$$\begin{aligned}
\int_C x\,dy + x^2\,dx &= -\iint_R 2x - 0\,dA \\
&= -\int_0^2 \int_{2x}^4 2x\,dy\,dx \\
&= -\int_0^2 2xy\Big|_{2x}^4\,dx \\
&= -\int_0^2 2x(4) - 2x(2x)\,dx \\
&= -\int_0^2 8x - 4x^2\,dx \\
&= \frac{4}{3}x^3 - 4x^2\Big|_0^2 \\
&= \frac{4}{3}(2)^3 - 4(2)^2
\end{aligned}$$

4. Let Σ be the portion of the plane $x + z = 4$ in the first octant between $y = 3$ and $y = 5$. If the mass density at a point is $f(x, y, z) = yz$ write down an iterated integral corresponding to the mass of Σ. **Do Not Evaluate This Integral.** [20 pts]

Solution:

We parametrize the surface Σ by:

$$\mathbf{r}(x,y) = x\,\mathbf{i} + y\,\mathbf{j} + (4-x)\,\mathbf{k}$$
$$\left.\begin{array}{c} 0 \le x \le 4 \\ 3 \le y \le 5 \end{array}\right\} R$$

We then have:

$$\begin{aligned} \mathbf{r}_x &= 1\,\mathbf{i} + 0\,\mathbf{j} - 1\,\mathbf{k} \\ \mathbf{r}_y &= 0\,\mathbf{i} + 1\,\mathbf{j} + 0\,\mathbf{k} \\ \mathbf{r}_x \times \mathbf{r}_y &= 1\,\mathbf{i} + 0\,\mathbf{j} + 1\,\mathbf{k} \\ ||\mathbf{r}_x \times \mathbf{r}_y|| &= \sqrt{2} \end{aligned}$$

Then the mass is:

$$\iint_{\Sigma} yz\,dS = \iint_{R} \sqrt{2}y(4-x)\,dA = \int_0^4 \int_3 \sqrt{2}y(4-x)\,dy\,dx$$

5. Let Σ be the intersection of the cylinder $x^2 + z^2 = 4$ with the plane $x + z = 4$ with counterclockwise orientation when viewed from the positive y-axis. Apply Stokes' Theorem to the integral $\int_C xz\,dx + x\,dy + y^2\,dz$ and proceed until you have an iterated double integral. [20 pts]
Do Not Evaluate This Integral.

Solution:

We apply Stokes Theorem to get:

$$\begin{aligned}\int_C xz\,dx + x\,dy + y^2\,dz &= \iint_\Sigma \left[(2y - 0)\,\mathbf{i} - (0 - x)\,\mathbf{j} + (1 - 0)\,\mathbf{k}\right] \cdot \mathbf{n}\,dS \\ &= \iint_\Sigma \left[2y\,\mathbf{i} + x\,\mathbf{j} + 1\,\mathbf{k}\right] \cdot \mathbf{n}\,dS\end{aligned}$$

Where Σ is the portion of the plane inside the cylinder with orientation towards the positve y-axis.

We parametrize the surface Σ by:

$$\mathbf{r}(r,\theta) = r\cos\theta\,\mathbf{i} + (4 - r\sin\theta)\,\mathbf{j} + r\sin\theta\,\mathbf{k}$$

$$\left.\begin{matrix} 0 \le \theta \le 2\pi \\ 0 \le r \le 2 \end{matrix}\right\} R$$

We then have:

$$\begin{aligned}\mathbf{r}_r &= \cos\theta\,\mathbf{i} - \sin\theta\,\mathbf{j} + r\sin\theta\,\mathbf{k} \\ \mathbf{r}_\theta &= -r\sin\theta\,\mathbf{i} - r\cos\theta\,\mathbf{j} + r\cos\theta\,\mathbf{k} \\ \mathbf{r}_r \times \mathbf{r}_\theta &= 0\,\mathbf{i} - r\,\mathbf{j} - r\,\mathbf{k}\end{aligned}$$

This is opposite to Σ's orientation so we introduce a minus. Thus:

$$\begin{aligned}\iint_\Sigma \left[2y\,\mathbf{i} + x\,\mathbf{j} + 1\,\mathbf{k}\right] \cdot \mathbf{n}\,dS &= -\iint_R \left[2(4 - r\sin\theta)\,\mathbf{i} + (r\cos\theta)\,\mathbf{j} + 1\,\mathbf{k}\right] \cdot \left[0\,\mathbf{i} - r\,\mathbf{j} - r\,\mathbf{k}\right]\,dA \\ &= -\int_0^{2\pi}\int_0^2 -r^2\cos\theta - r\,dr\,d\theta\end{aligned}$$