MATH 241 Sections 03** Exam 4 Spring 2021

Exam Submission:

1. Submit this exam to Gradescope.
2. Tag your problems!
3. You may print the exam, write on it, scan and upload.
4. Or you may just write on it on a tablet and upload.
5. Or you are welcome to write the answers on a separate piece of paper if other options don't appeal to you, then scan and upload.

Exam Rules:

1. You may ask me for clarification on questions but you may not ask me for help on questions!
2. You are permitted to use any non-interactive resources. This includes books, static pages on the internet, your notes, and YouTube videos.
3. You are not permitted to use any interactive resources. This includes your friends, your friends' friends, your calculator, Matlab, Wolfram Alpha, and online chat groups.
   Exception: Calculators are fine for basic arithmetic.
4. If you are unsure about whether a resource is considered "interactive" simply ask me and I'll let you (and everyone) know.
5. Petting small animals for stress relief is acceptable and is not considered an "interactive resource".

Work Shown:

1. Show all work as appropriate for and using techniques learned in this course.
2. Any pictures, work and scribbles which are legible and relevant will be considered for partial credit.

1. **Instruction:**

Let A be the sum of all of the even digits in your UID.
If your $A = 0$ then use $A = 7$ instead.

Write down your UID and the value(s) and mark them clearly. In the problem below, replace them by the appropriate value(s) before proceeding.

Suppose C is a piece of wire in the shape of a parabola with equation $y = Ax^2$ for $0 \leq x \leq A$. Here x, y are in centimeters. If the electrical charge density at (x, y) is given by $f(x, y) = A^2 x$ coulombs per centimeter calculate the total electrical charge on the wire. [15 pts]

You Should Evaluate Your Resulting Integral!

Solution:

We have $\bar{\boldsymbol{r}}(t) = t\,\hat{\imath} + At^2\,\hat{\jmath}$ for $0 \leq t \leq A$. The total electrical charge is:

$$\begin{aligned}
\text{Charge} &= \int_C A^2 x\,ds \\
&= \int_0^A A^2 t \|1\,\hat{\imath} + 2At\,\hat{\jmath}\|\,dt \\
&= \int_0^A A^2 t\sqrt{1 + 4A^2t^2}\,dt \\
&= \frac{1}{12}\left(1 + 4A^2t^2\right)^{3/2}\Big|_0^A \\
&= \frac{1}{12}\left(1 + 4A^4\right)^{3/2} - \frac{1}{12}
\end{aligned}$$

2. **Instruction:**

Let B be the product of the leftmost three nonzero digits of your UID.

Write down your UID and the value(s) and mark them clearly. In the problem below, replace them by the appropriate value(s) before proceeding.

Suppose an object follows the clockwise triangular path from $(0,0)$ to $(B,1)$ to $(1,0)$ and back to $(0,0)$. Calculate the amount of work done on the object by the vector field $\bar{\boldsymbol{F}}(x,y) = 0.5y^2\,\hat{\imath} - xy\,\hat{\jmath}$. [15 pts]

$$\boxed{\textbf{You Should Evaluate Your Resulting Integral!}}$$

Solution:

We are looking for $\int_C (0.5y^2\,\hat{\imath} - xy\,\hat{\jmath}) \cdot d\bar{\boldsymbol{r}}$. By Green's Theorem we have:

$$\int_C (0.5y^2\,\hat{\imath} - xy\,\hat{\jmath}) \cdot d\bar{\boldsymbol{r}} = -\iint_R -2y\,dA$$

If we parametrize R as a horizontally simple region then the left function is $x = By$ and the right function is $x = (B-1)y+1$ and so the result is as follows. Notice (this was unplanned but funny) that the result doesn't depend on B.

$$\begin{aligned}
\int_C (0.5y^2\,\hat{\imath} - xy\,\hat{\jmath}) \cdot \bar{\boldsymbol{r}} &= -\iint_R -2y\,dA \\
&= \int_R 2y\,dA \\
&= \int_0^1 \int_{By}^{(B-1)y+1} 2y\,dx\,dy \\
&= \int_0^1 2xy\Big|_{By}^{(B-1)y+1}\,dy \\
&= \int_0^1 2((B-1)y+1)y - 2(By)y\,dy \\
&= \int_0^1 -2y^2 + 2y\,dy \\
&= -\frac{2}{3}y^3 + y^2\Big|_0^1 \\
&= -\frac{2}{3} + 1 \\
&= \frac{1}{3}
\end{aligned}$$

3. **Instruction:**

Let C be the sum of the first four digits of your UID.
Let D be the sum of the last four digits of your UID.

Write down your UID and the value(s) and mark them clearly. In the problem below, replace them by the appropriate value(s) before proceeding.

Suppose Σ is the portion of the plane $Cx + Dy + z = CD$ in the first octant. Find the surface area of Σ. [15 pts]

You Should Evaluate Your Resulting Integral!

Solution:

We parametrize Σ by:

$$\bar{\boldsymbol{r}}(x, y) = x\,\hat{\imath} + y\,\hat{\jmath} + (CD - Cx - Dy)\,\hat{k}$$
$$0 \leq x \leq D$$
$$0 \leq y \leq C - \tfrac{C}{D}x$$

We then have:

$$\begin{aligned}
\bar{\boldsymbol{r}}_x &= 1\,\hat{\imath} + 0\,\hat{\jmath} - C\,\hat{k} \\
\bar{\boldsymbol{r}}_y &= 0\,\hat{\imath} + 1\,\hat{\jmath} - D\,\hat{k} \\
\bar{\boldsymbol{r}}_x \times \bar{\boldsymbol{r}}_y &= C\,\hat{\imath} + D\,\hat{\jmath} + 1k \\
\|\bar{\boldsymbol{r}}_x \times \bar{\boldsymbol{r}}_y\| &= \sqrt{C^2 + D^2 + 1}
\end{aligned}$$

The surface area is then:

$$\begin{aligned}
\text{SA} &= \iint_\Sigma 1\,dS \\
&= \iint_R \sqrt{C^2 + D^2 + 1}\,dA \\
&= \sqrt{C^2 + D^2 + 1} \int_R 1\,dA \\
&= \sqrt{C^2 + D^2 + 1} \int_0^D \int_0^{C - \frac{C}{D}x} 1\,dy\,dx \\
&= \sqrt{C^2 + D^2 + 1} \int_0^D x\Big|_0^{C - \frac{C}{D}x}\,dx \\
&= \sqrt{C^2 + D^2 + 1} \int_0^D C - \frac{C}{D}x\,dx \\
&= \sqrt{C^2 + D^2 + 1} \left(Cx - \frac{C}{2D}x^2\right)\Big|_0^D \\
&= \sqrt{C^2 + D^2 + 1} \left(CD - \frac{C}{2D}D^2\right) \\
&= \frac{1}{2}CD\sqrt{C^2 + D^2 + 1}
\end{aligned}$$

4. Let C be any curve from $(1,2,3)$ to $(4,3,-2)$. Evaluate and simplify: [10 pts]

$$\int_C \left(\frac{2x}{y} + ze^{xz}\right) dx + \left(-\frac{x^2}{y^2}\right) dy + xe^{xz}\, dz$$

Solution:

The vector field is conservative with potential function:

$$f(x,y,z) = \frac{x^2}{y} + e^{xz}$$

Thus by the Fundamental Theorem of Line Integrals we have:

$$\begin{aligned}\int_C \left(\frac{2x}{y} + ze^{xz}\right) dx + \left(-\frac{x^2}{y^2}\right) dy + xe^{xz}\, dz &= f(4,3,-2) - f(1,2,3) \\ &= \left[\frac{4^2}{3} + e^{(4)(-2)}\right] - \left[\frac{1^2}{2} + e^{(1)(3)}\right]\end{aligned}$$

5. Suppose Σ is the part of the cone $y = 3 - \sqrt{x^2+z^2}$ with $y \geq 0$ along with the base $x^2 + z^2 \leq 9$ on the xz-plane. Suppose Σ has inwards orientation. [10 pts]

Evaluate:

$$\iint_\Sigma (4x\,\hat{\imath} + 3y\,\hat{\jmath} - z\,\hat{k}) \cdot \bar{\boldsymbol{n}}\, dS$$

Solution:

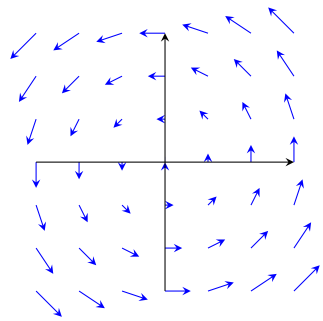
Since the surface is closed around the solid cone D we can use the Divergence Theorem with negation to get:

$$\iint_\Sigma (4x\,\hat{\imath} + 3y\,\hat{\jmath} - z\,\hat{k}) \cdot \bar{\boldsymbol{n}}\, dS = -\iiint_D 6\, dV$$

This is just -6 times the volume of the cone, hence:

$$-6\left(\frac{1}{3}\pi(3)^2(3)\right) = -54\pi$$

6. Explain briefly why the following vector field is not conservative: [5 pts]



Solution:

Imagine an object following a circular path around the origin and ending at the start point. The circular nature of the vector field indicates that work will be done, meaning the integral is not zero.

7. Suppose we have $f(x, y, z) = x^2y - xz$. Only one of the following makes sense. Circle the one that does and then calculate it. You do not need to justify how you made your choice. [10 pts]

$$\nabla \cdot (\nabla f) \qquad \text{and} \qquad \nabla(\nabla \cdot f)$$

Solution:

The one on the left makes sense and it equals:

$$\nabla \cdot (\nabla f) = \nabla \cdot ((2xy - z)\,\hat{\imath} + x^2\,\hat{\jmath} - x\,\hat{k}\,) = 2y$$

8. Suppose C is the intersection curve of the plane $x + 3y = 12$ with the cylinder $y^2 + z^2 = 9$ with counterclockwise orientation when viewed towards the origin from way out on the positive x-axis. Apply Stokes' Theorem to the integral [20 pts]

$$\int_C (x+z)\,dx + x^2\,dy + xy\,dz$$

and proceed until you have an iterated double integral.

$\boxed{\textbf{You Should Not Evaluate Your Resulting Integral!}}$

Solution:

By Stokes' Theorem we have:

$$\int_C (x+z)\,dx + x^2\,dy + xy\,dz = \iint_\Sigma \left[x\,\hat{\imath} - (y-1)\,\hat{\jmath} + 2x\,\hat{k}\right] \cdot \bar{\boldsymbol{n}}\,dS$$

Where Σ is the part of the plane inside the cylinder, oriented to the right (positive $\hat{\jmath}$) and forward (positive $\hat{k}$).

We parametrize Σ:

$$\bar{\boldsymbol{r}}(\theta, r) = (12 - 3r\cos\theta)\,\hat{\imath} + r\cos\theta\,\hat{\jmath} + r\sin\theta\,\hat{k}$$
$$0 \le \theta \le 2\pi$$
$$0 \le r \le 3$$

Then:

$$\begin{aligned}
\bar{\boldsymbol{r}}_\theta &= 3r\sin\theta\,\hat{\imath} - r\sin\theta\,\hat{\jmath} + r\cos\theta\,\hat{k} \\
\bar{\boldsymbol{r}}_r &= -3\cos\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath} + \sin\theta\,\hat{k} \\
\bar{\boldsymbol{r}}_\theta \times \bar{\boldsymbol{r}}_r &= -r\,\hat{\imath} - 3r\,\hat{\jmath} + 0\,\hat{k}
\end{aligned}$$

Which opposes Σ's orientation. Thus we have:

$\iint_\Sigma \left[x\,\hat{\imath} - (y-1)\,\hat{\jmath} + 2x\,\hat{k}\right] \cdot \bar{\boldsymbol{n}}\,dS$

$$\begin{aligned}
&= -\iint_R \left[(12 - 3r\cos\theta)\,\hat{\imath} - (r\cos\theta - 1)\,\hat{\jmath} + 2(12 - 3r\cos\theta)\,\hat{k}\right] \cdot \bar{\boldsymbol{n}}\,dS \\
&= -\int_R \left[(12 - 3r\cos\theta)\,\hat{\imath} - (r\cos\theta - 1)\,\hat{\jmath} + 2(12 - 3r\cos\theta)\,\hat{k}\right] \cdot (-r\,\hat{\imath} - 3r\,\hat{\jmath} + 0\,\hat{k})\,dA \\
&= -\int_R (12 - 3r\cos\theta)(-r) - (r\cos\theta - 1)(-3r) + 2(12 - 3r\cos\theta)(0)\,dA \\
&= -\int_0^{2\pi}\int_0^3 (12 - 3r\cos\theta)(-r) - (r\cos\theta - 1)(-3r)\,dr\,d\theta
\end{aligned}$$