**MATH241 Exam 4 Spring 2022 (Justin Wyss-Gallifent)**
**May the 4th be with you!**

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| **Directions:** Please do work in the spaces provided and do not spill over to other pages - the exams will be scanned into Gradescope and auto-tagged this way. Show work and use only methods taught in this course and show work as indicated. No calculators or other devices permitted. Numerical answers do not need to be simplified. Good luck! |

Extra Credit: Write the date, time, building and room of your final exam. [$\leq$ 2 pts]
**Solution:**

1. Write the word TRUE or FALSE in the box to the right. No justification is required. [10 pts]

   (a) If $C$ is closed and $\mathbf{F}$ is conservative then $\int_C \mathbf{F} \cdot d\mathbf{r} \neq 0$. $\boxed{\text{FALSE}}$

   (b) If $\mathbf{F}$ is conservative then $\nabla \times \mathbf{F} = \mathbf{0}$. $\boxed{\text{TRUE}}$

   (c) Green's Theorem applies if $C$ is not closed. $\boxed{\text{FALSE}}$

   (d) We only have one way to evaluate $\int_C f\,ds$. $\boxed{\text{TRUE}}$

   (e) The expression $\nabla \times f(x, y, z)$ makes sense. $\boxed{\text{FALSE}}$

2. Let $C$ be the curve parametrized by $\mathbf{r}(t) = 3\cos t\,\hat{\boldsymbol{\imath}} + 4\sin t\,\hat{\boldsymbol{\jmath}}$ for $0 \leq t \leq \frac{\pi}{2}$. Without integrating, explain how you know that $\int_C 1\,ds > 5$. [10 pts]
   Hint: Sketch it!

   **Solution:**

   The integral is measuring the length of $C$. Since the curve goes from $(3, 0)$ to $(0, 4)$ and is not a straight line, it is more than $\sqrt{3^2 + 4^2} = 5$ units long.

3. Let $\Sigma$ be the part of the plane $2x+y+z=4$ in the first octant. If the mass density at a point is given by $f(x,y,z)=x$, write down and evaluate the integral for the mass of $\Sigma$. [20 pts]

**You Should Evaluate Your Result!**

**Solution:**

We parametrize $\Sigma$ by:

$$\begin{gathered}\mathbf{r}(x,y) = x\,\hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}} + (4-2x-y)\,\hat{\boldsymbol{k}} \\ 0 \le x \le 2 \\ 0 \le y \le 4-2x\end{gathered}$$

We have:

$$\begin{aligned}\mathbf{r}_x &= 1\,\hat{\boldsymbol{\imath}} + 0\,\hat{\boldsymbol{\jmath}} - 2\,\hat{\boldsymbol{k}} \\ \mathbf{r}_y &= 0\,\hat{\boldsymbol{\imath}} + 1\,\hat{\boldsymbol{\jmath}} - 1\,\hat{\boldsymbol{k}} \\ \mathbf{r}_x \times \mathbf{r}_y &= 2\,\hat{\boldsymbol{\imath}} + 1\,\hat{\boldsymbol{\jmath}} + 1\,\hat{\boldsymbol{k}} \\ ||\mathbf{r}_x \times \mathbf{r}_y|| &= \sqrt{6}\end{aligned}$$

Then:

$$\begin{aligned}\text{Mass} &= \iint_\Sigma x\,dS \\ &= \iint_R x\sqrt{6}\,dA \\ &= \sqrt{6}\int_0^2 \int_0^{4-2x} x\,dy\,dx \\ &= \sqrt{6}\int_0^2 xy\Big|_0^{4-2x}\,dx \\ &= \sqrt{6}\int_0^2 x(4-2x) - x(0)\,dx \\ &= \sqrt{6}\left(2x^2 - \frac{2}{3}x^3\right)\Big|_0^2 \\ &= \sqrt{6}\left(8 - \frac{2}{3}(8)\right)\end{aligned}$$

4. Suppose $f(x,y,z) = \frac{x}{y} - xz^2$ and $C$ is the straight line segment from $(1,2,3)$ to $(8,10,100)$. [5 pts]
Evaluate the line integral:

$$\int_C (\nabla f(x,y,z)) \cdot d\mathbf{r}$$

**You Should Evaluate Your Result!**

**Solution:**

By the FTOLI we have:

$$\int_C (\nabla f(x,y,z)) \cdot d\mathbf{r} = f(8,10,100) - f(1,2,3) = \left(\frac{8}{10} - (8)100^2\right) - \left(\frac{1}{2} - (1)3^2\right)$$

5. Use Green's Theorem to evaluate the line integral $\int_C 4y\,dx + 6x\,dy$ where $C$ is the curve parametrized by $\mathbf{r}(t) = (1+\cos t)\,\hat{\boldsymbol{\imath}} + (2+\sin t)\,\hat{\boldsymbol{\jmath}}$ for $0 \le t \le 2\pi$. [15 pts]

**You Should Evaluate Your Result!**

**Solution:**

If $R$ is the region inside $C$ then $R$ is a disk of radius 1 centered at $(1,2)$. Then we have:

$$\int_C 4y\,dx + 6x\,dy = \iint_R 6 - 4\,dA = 2\int_R 1\,dA = 2(\text{Area}) = 2\pi(1)^2$$

6. Let $\Sigma$ be the part of $z = x^2 + y^2$ inside $z = 8 - x^2 - y^2$ as well as the part of $z = 8 - x^2 - y^2$ inside $z = x^2 + y^2$, oriented outwards. Apply the Divergence Theorem to construct an iterated triple integral in cylindrical coordinates for the following: [15 pts]

$$\iint_\Sigma (xy\,\hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}} + xz\,\hat{\boldsymbol{k}}) \cdot \mathbf{n}\, dS$$

**You Should Not Evaluate Your Result!**

**Solution:**

The surface $\Sigma$ is the boundary of $D$ where $D$ is the solid between the two.

If we set them equal we get $x^2 + y^2 = 8 - x^2 - y^2$ which yields a disk of radius 2.

Thus this is the solid defined on the disk $x^2 + y^2 \le 4$ with floor $z = x^2 + y^2$ and ceiling $z = 8 - x^2 - y^2$.

We then have:

$$\iint_\Sigma (xy\,\hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}} + xz\,\hat{\boldsymbol{k}}) \cdot \mathbf{n}\, d = \iiint_D (y + 1 + x)\, dV = \int_0^{2\pi} \int_0^2 \int_{r^2}^{8-r^2} (r\sin\theta + 1 + r\cos\theta)\, r\, dz\, dr\, d\theta$$

7. Let $\Sigma$ be the part of the vertical plane $x + 2y = 4$ in the first octant and below $z = 5$. Let $C$ be the edge of $\Sigma$ with clockwise orientation when viewed from the first octant looking towards the origin. Consider the integral: [25 pts]

$$\int_C z^2\,dx + x\,dy + z\,dz$$

Use Stokes' Theorem to rewrite the line integral as a surface integral and then as an iterated double integral.

$\boxed{\textbf{You Should Not Evaluate Your Result!}}$

**Solution:**

We see that $C$ is the edge of the surface $\Sigma$ where $\Sigma$ is given. By Stokes' Theorem we then have:

$$\begin{aligned}\int_C z^2\,dx + x\,dy + z\,dz &= \iint_\Sigma ((0-0)\,\hat{\boldsymbol{\imath}} - (0-2z)\,\hat{\boldsymbol{\jmath}} + (1-0)\,\hat{\boldsymbol{k}}\,)\cdot\mathbf{n}\,dS \\ &= \iint_\Sigma (0\,\hat{\boldsymbol{\imath}} + 2z\,\hat{\boldsymbol{\jmath}} + 1\,\hat{\boldsymbol{k}}\,)\cdot\mathbf{n}\,dS\end{aligned}$$

Here $\Sigma$ has orientation out into the first octant, induced by $C$.

We then parameterize $\Sigma$ by:

$$\begin{gathered}\mathbf{r}(y,z) = (4-2y)\,\hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}} + z\,\hat{\boldsymbol{k}} \\ 0 \le y \le 2 \\ 0 \le z \le 6\end{gathered}$$

We have:

$$\begin{aligned}\mathbf{r}_y &= -2\,\hat{\boldsymbol{\imath}} + 1\,\hat{\boldsymbol{\jmath}} + 0\,\hat{\boldsymbol{k}} \\ \mathbf{r}_z &= 0\,\hat{\boldsymbol{\imath}} + 0\,\hat{\boldsymbol{\jmath}} + 1\,\hat{\boldsymbol{k}} \\ \mathbf{r}_y \times \mathbf{r}_z &= 1\,\hat{\boldsymbol{\imath}} + 2\,\hat{\boldsymbol{\jmath}} + 0\,\hat{\boldsymbol{k}}\end{aligned}$$

These do not match the orientation of $\Sigma$ and hence, letting $R$ represent the set of inequalities:

$$\begin{aligned}\iint_\Sigma (0\,\hat{\boldsymbol{\imath}} + 2z\,\hat{\boldsymbol{\jmath}} + 1\,\hat{\boldsymbol{k}}\,)\cdot\mathbf{n}\,dS &= -\iint_R (0\,\hat{\boldsymbol{\imath}} + 2z\,\hat{\boldsymbol{\jmath}} + 1\,\hat{\boldsymbol{k}}\,)\cdot(1\,\hat{\boldsymbol{\imath}} + 2\,\hat{\boldsymbol{\jmath}} + 0\,\hat{\boldsymbol{k}}\,)\,dA \\ &= -\iint_R 4z\,dA \\ &= -\int_0^2\int_0^5 4z\,dz\,dy\end{aligned}$$