MATH241 Spring 2023 Exam 4 (Justin W-G) Solutions

Name (Neatly)	
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Instructions:

1. Please do all problems on the pages and in the spaces provided. This exam will be scanned into Gradescope and if your answers are not in the correct locations they will not be found or graded!
2. Only simplify Calculus 3 related calculations.

Extra Credit: Write the date, time, building and room of your final exam in the table below. If taking with ADS, write "ADS" in the Building and Room spaces. [$\leq$ 2 pts]

Solution:

Date	Time	Building	Room

1. Write T for True or F for False in the box to the right. No justification is required. Unreadable or ambiguous letters will be marked as incorrect. [10 pts]

Solution:

Statement	T or F
$\iint_\Sigma 1\,dS$ yields surface area.	T
The line integral of a conservative vector always equals zero.	F
Stokes' Theorem applies to integrals of the form $\iiint_D f(x,y,z)\,dV$.	F
The Fundamental Theorem of Line Integrals requires a conservative vector field.	T
The Divergence Theorem got its name because the answer always diverges to ∞.	F

2. Let Σ be the part of $x + y = 2$ in the first octant with $0 \leq z \leq 5$. The following integral can be evaluated without any integration. Briefly explain and give the answer. [5 pts]

$$\iint_\Sigma 4\,dS$$

NOTE: Zero points will be given for actually integrating!

Solution:

The integral equals 4 times the surface area of the rectangle. The rectangle is $5 \times \sqrt{8}$ so the result is $4(5)(\sqrt{8})$.

3. Prove that the following vector field is not conservative: [5 pts]

$$\bar{\boldsymbol{F}}(x,y,z) = xz\,\hat{\boldsymbol{\imath}} + (z-y)\,\hat{\boldsymbol{\jmath}} + y\,\hat{\boldsymbol{k}}$$

Solution:

We have:

$$\nabla \times \bar{\boldsymbol{F}} = (1-1)\,\hat{\boldsymbol{\imath}} - (0-x)\,\hat{\boldsymbol{\jmath}} + (0-0)\,\hat{\boldsymbol{k}} \neq \bar{\boldsymbol{0}}$$

4. The following integrals can be calculated using a theorem followed by an easy calculation. For each, state the theorem and evaluate.

(a) Let C be parameterized by $\bar{\boldsymbol{r}}(t) = t^2\,\hat{\boldsymbol{\imath}} + t\cos t\,\hat{\boldsymbol{\jmath}}$ for $0 \le t \le \pi$. [6 pts]

$$\int_C (1+y)\,dx + x\,dy$$

Solution:

Since the VF is conservative with potential $f(x,y) = x + xy$ we can use the FTOLI. We have starting point $\bar{\boldsymbol{r}}(0) = 0\,\hat{\boldsymbol{\imath}} + 0\,\hat{\boldsymbol{\jmath}}$ or $(0,0)$ and ending point $\bar{\boldsymbol{r}}(\pi) = \pi^2\,\hat{\boldsymbol{\imath}} - \pi\,\hat{\boldsymbol{\jmath}}$ or $(\pi^2, -\pi)$ and hence the result is:

$$f(\pi^2, \pi) - f(0,0) = \pi^2 + \pi^2(-\pi) = \pi^2 - \pi^3$$

(b) Let C be the clockwise triangle with corners $(0,0)$, $(2,2)$, and $(6,0)$. [6 pts]

$$\int_C 2xy\,dx + (x^2 - 7x)\,dy$$

Solution:

Since C is closed around the filled-in triangle R with the same vertices we can use GT but negate because it's clockwise:

$$\int_C 2xy\,dx + (x^2 - 7x)\,dy = -\iint_R 2x - 7 - 2x\,dA = -7(\text{area}) = 7\left(\frac{1}{2}(6)(2)\right) = 42$$

(c) Let Σ be the part of the cone $z = 2 - \sqrt{x^2 + y^2}$ above the xy-plane along with the disk $x^2 + y^2 \le 4$ in the xy plane. Assume Σ is oriented outwards. [8 pts]

$$\iint_\Sigma (5x\,\hat{\boldsymbol{\imath}} + 7x\,\hat{\boldsymbol{\jmath}} - 10z\,\hat{\boldsymbol{k}}) \cdot \bar{\boldsymbol{n}}\,dS$$

Solution:

Since Σ is the boundary of the solid cone D we can use the DT:

$$\iint_\Sigma (5x\,\hat{\boldsymbol{\imath}} + 7x\,\hat{\boldsymbol{\jmath}} - 10z\,\hat{\boldsymbol{k}}) \cdot \bar{\boldsymbol{n}}\,dS = \iiint_V 4 + 0 - 10\,dV = -6(\text{volume}) = -6\left(\frac{1}{3}\pi(2)^2(4)\right)$$

5. Let C be the line segment from $(0,1)$ to $(8,3)$. Evaluate the line integral: [10 pts]

$$\int_C (x\,\hat{\boldsymbol{\imath}} + xy\,\hat{\boldsymbol{\jmath}}) \cdot d\bar{\boldsymbol{r}}$$

YOU MUST EVALUATE THIS INTEGRAL!

Solution:

We parameterize the line with $\bar{\boldsymbol{r}}(t) = 8t\,\hat{\boldsymbol{\imath}} + (1+2t)\,\hat{\boldsymbol{\jmath}}$ for $0 \le t \le 1$ and then $\bar{\boldsymbol{r}}'(t) = 8\,\hat{\boldsymbol{\imath}} + 2\,\hat{\boldsymbol{\jmath}}$ and so:

$$\begin{aligned}
\int_C (x\,\hat{\boldsymbol{\imath}} + xy\,\hat{\boldsymbol{\jmath}}) \cdot d\bar{\boldsymbol{r}} &= \int_0^1 (8t\,\hat{\boldsymbol{\imath}} + 8t(1+2t)\,\hat{\boldsymbol{\jmath}}) \cdot (8\,\hat{\boldsymbol{\imath}} + 2\,\hat{\boldsymbol{\jmath}})\,dt \\
&= \int_0^1 80t + 32t^2\,dt \\
&= 40t^2 + \frac{32}{3}t^3\Big|_0^1 \\
&= 40 + \frac{32}{3}
\end{aligned}$$

6. Let Σ be the part of the parabolic sheet $y = x^2$ between $x = -1$ and $x = 2$ and with $0 \leq z \leq 7$. [15 pts]
Write down an iterated double integral for the surface area of Σ.

DO NOT EVALUATE!

Solution:

We parameterize the surface by $\bar{\boldsymbol{r}}(x, z) = x\,\hat{\boldsymbol{\imath}} + x^2\,\hat{\boldsymbol{\jmath}} + z\,\hat{\boldsymbol{k}}$ with $-1 \leq x \leq 2$ and $0 \leq z \leq 7$ and then:

$$\begin{aligned}
\bar{\boldsymbol{r}}_x &= 1\,\hat{\boldsymbol{\imath}} + 2x\,\hat{\boldsymbol{\jmath}} + 0\,\hat{\boldsymbol{k}} \\
\bar{\boldsymbol{r}}_z &= 0\,\hat{\boldsymbol{\imath}} + 0\,\hat{\boldsymbol{\jmath}} + 1\,\hat{\boldsymbol{k}} \\
\bar{\boldsymbol{r}}_x \times \bar{\boldsymbol{r}}_z &= 2x\,\hat{\boldsymbol{\imath}} + 1\,\hat{\boldsymbol{\jmath}} + 0\,\hat{\boldsymbol{k}} \\
||\bar{\boldsymbol{r}}_x \times \bar{\boldsymbol{r}}_z|| &= \sqrt{4x^2 + 1}
\end{aligned}$$

So the surface area is:

$$\iint_\Sigma 1\,dS = \iint_R \sqrt{4x^2 + 1}\,dA = \int_0^7 \int_{-1}^2 \sqrt{4x^2 + 1}\,dx\,dz$$

7. Let Σ be the part of the plane $2x + 3y + z = 6$ in the first octant and oriented upwards. Write down an iterated double integral for the rate of flow of $\bar{\boldsymbol{F}}(x, y, z) = 0\,\hat{\boldsymbol{\imath}} + 0\,\hat{\boldsymbol{\jmath}} + z\,\hat{\boldsymbol{k}}$ through Σ. [15 pts]

DO NOT EVALUATE!

Solution:

We parameterize the surface by $\bar{\boldsymbol{r}}(x, y) = x\,\hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}} + (6 - 2x - 3y)\,\hat{\boldsymbol{k}}$ with $0 \leq x \leq 3$ and $0 \leq y \leq 2 - \frac{2}{3}x$ and then:

$$\begin{aligned}
\bar{\boldsymbol{r}}_x &= 1\,\hat{\boldsymbol{\imath}} + 0\,\hat{\boldsymbol{\jmath}} - 2\,\hat{\boldsymbol{k}} \\
\bar{\boldsymbol{r}}_y &= 0\,\hat{\boldsymbol{\imath}} + 1\,\hat{\boldsymbol{\jmath}} - 3\,\hat{\boldsymbol{k}} \\
\bar{\boldsymbol{r}}_x \times \bar{\boldsymbol{r}}_y &= 2\,\hat{\boldsymbol{\imath}} + 3\,\hat{\boldsymbol{\jmath}} + 1\,\hat{\boldsymbol{k}}
\end{aligned}$$

These match Σ's upwards orientation and so:

$$\begin{aligned}
\text{Flow} &= \iint_{\Sigma} (0\,\hat{\boldsymbol{\imath}} + 0\,\hat{\boldsymbol{\jmath}} + z\,\hat{\boldsymbol{k}}) \cdot \bar{\boldsymbol{n}}\,dS \\
&= +\iint_{R} (0\,\hat{\boldsymbol{\imath}} + 0\,\hat{\boldsymbol{\jmath}} + (6 - 2x - 3y)\,\hat{\boldsymbol{k}}) \cdot (2\,\hat{\boldsymbol{\imath}} + 3\,\hat{\boldsymbol{\jmath}} + 1\,\hat{\boldsymbol{k}})\,dA \\
&= \int_0^3 \int_0^{2 - \frac{2}{3}x} 6 - 2x - 3y\,dy\,dx
\end{aligned}$$

8. Let C be the intersection of $x^2 + z^2 = 9$ with $x + y = 5$ and with counterclockwise orientation when viewed from the positive y-axis. Use Stokes' Theorem to obtain an iterated double integral which calculates: [20 pts]

$$\int_C x\,dx + xz\,dy + xy\,dz$$

DO NOT EVALUATE!

Solution:

Stokes' Theorem states that:

$$\int_C x\,dx{+}xz\,dy{+}xy\,dz = \iint_\Sigma \left[(x-x)\,\hat{\boldsymbol{\imath}} - (y-0)\,\hat{\boldsymbol{\jmath}} + (z-0)\,\hat{\boldsymbol{k}}\right]\cdot\bar{\boldsymbol{n}}\,dS = \iint_\Sigma \left[0\,\hat{\boldsymbol{\imath}} - y\,\hat{\boldsymbol{\jmath}} + z\,\hat{\boldsymbol{k}}\right]\cdot\bar{\boldsymbol{n}}\,dS$$

Where Σ is the part of the plane $x + y = 5$ inside the cylinder $x^2 + y^2 = 9$ with orientation in the positive x and y directions.

We parameterize Σ by $\bar{\boldsymbol{r}}(\theta, r) = r\cos\theta\,\hat{\boldsymbol{\imath}} + (5 - r\cos\theta)\,\hat{\boldsymbol{\jmath}} + r\sin\theta\,\hat{\boldsymbol{k}}$ with $0 \le \theta \le 2\pi$ and $0 \le r \le 3$ and then:

$$\begin{aligned}
\bar{\boldsymbol{r}}_\theta &= -r\sin\theta\,\hat{\boldsymbol{\imath}} + r\sin\theta\,\hat{\boldsymbol{\jmath}} + r\cos\theta\,\hat{\boldsymbol{k}} \\
\bar{\boldsymbol{r}}_r &= \cos\theta\,\hat{\boldsymbol{\imath}} - \cos\theta j + \sin\theta\,\hat{\boldsymbol{k}} \\
\bar{\boldsymbol{r}}_\theta \times \bar{\boldsymbol{r}}_r &= r\,\hat{\boldsymbol{\imath}} + r\,\hat{\boldsymbol{\jmath}} + 0\,\hat{\boldsymbol{k}}
\end{aligned}$$

These match Σ's orientation and so:

$$\begin{aligned}
\int_C x\,dx + xz\,dy + xy\,dz &= \iint_\Sigma \left[0\,\hat{\boldsymbol{\imath}} - y\,\hat{\boldsymbol{\jmath}} + z\,\hat{\boldsymbol{k}}\right]\cdot\bar{\boldsymbol{n}}\,dS \\
&= +\iint_R \left[0\,\hat{\boldsymbol{\imath}} - (5 - r\cos\theta)\,\hat{\boldsymbol{\jmath}} + r\sin\theta\,\hat{\boldsymbol{k}}\right]\cdot(r\,\hat{\boldsymbol{\imath}} + r\,\hat{\boldsymbol{\jmath}} + 0\,\hat{\boldsymbol{k}})\,dA \\
&= \int_0^{2\pi}\int_0^3 -r(5 - r\cos\theta)\,dr\,d\theta
\end{aligned}$$