Math 241 Section 12.4: Curves and Associated Definitions
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1. A space curve (a curve) is the range of a VVF. Really it's just the curve we draw. A curve can be discussed without explicitly giving an $\mathbf{r}(t)$. A choice of $\mathbf{r}(t)$ is defined as a parametrization of the curve and one curve can have many parametrizations.
   Example: A line.
   Example: A circle.

2. Associated Definitions

   (a) Closed:

   - A *closed parametrization* is a parametrization with the property that $\mathbf{r}(a) = \mathbf{r}(b)$ (it starts where it ends) but otherwise does not cross itself infinitely many times.
     Example: $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$ for $0 \leq t \leq 2\pi$.
     Example: If t goes any further like twice around then the curve crosses itself infinitely many times and that's a no-no.
   - A curve is *closed* if it has a closed parametrizations.
     Example: The circle again. There are lots of parametrizations of the circle but since there is one that's closed then the curve is closed.

   (b) Smooth:

   - A *smooth parametrization* has $\mathbf{r}'(t) \neq \mathbf{0}$ except it is permitted to be $\mathbf{0}$ at the endpoints, if it has endpoints. A good analogy is an ideal commute where your velocity is (probably) $\mathbf{0}$ at the start and end but you never need to stop on the way.
   - A curve is *smooth* if it has a smooth parametrization.

   (c) Piecewise Smooth:

   - A *piecewise smooth* parametrization is a parametrization in which you can break the t-range into pieces on which the parametrization is smooth. Your commute is probably more like this.
   - A curve is *piecewise smooth* if it has a piecewise smooth parametrization.

   Example: $\mathbf{r}(t) = (2t+1)\,\mathbf{i} + (3-t)\,\mathbf{j} + t\,\mathbf{j}$ for all t. Here $\mathbf{r}'(t) = 2\,\mathbf{i} - 1\,\mathbf{j} + 1\,\mathbf{k}$ which is never $\mathbf{0}$ hence smooth.
   Example: $\mathbf{r}(t) = t^2\,\mathbf{i} + t^3\,\mathbf{j} + 2\,\mathbf{k}$ on $[0,5]$. Here $\mathbf{r}'(t) = 2t\,\mathbf{i} + 3t^2\,\mathbf{j} + 0\,\mathbf{k}$ which is $\mathbf{0}$ at $t = 0$ but that's an endpoint so it's okay.
   Example: $\mathbf{r}(t)$ same as above but on $[-5,5]$. Here $t = 0$ is no longer an endpoint so it's not smooth. It is, however, piecewise smooth.
   Example: $\mathbf{r}(t) = t^{1/3}\,\mathbf{i}$ on $[-1,1]$. Here $\mathbf{r}'(t) = \frac{1}{3t^{2/3}}\,\mathbf{i}$ which is undefined at $t = 0$ hence neither smooth nor piecewise smooth.

   (d) Length: If C is piecewise smooth on $[a,b]$ then:

   $$\text{Length of } C = \int_a^b ||\mathbf{r}'(t)||\,dt$$

   Example: An easy one.

Note: Examples like 22,25 in the text are good because they require a sneaky factorization inside the square root before integration.