Math 241 Section 12.5: Tangents and Normals to Curves
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1. Intro:

   I discussed how sometimes we want unit vectors which are tangent to and normal to a curve, what "normal to a curve" might mean. We'll see why soon in this section.

2. The Tangent Vector:

   Defined
   $$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{||\mathbf{r}'(t)||}$$
   This one is usually pretty intuitive. Emphasized that it's length 1 and points in the direction of the curve.

3. The Normal Vector:

   Defined
   $$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{||\mathbf{T}'(t)||}$$
   We do this because $\mathbf{a}(t)$ is not normal (it contains both change in velocity in the direction of motion and perpendicular to it) and so by taking $\mathbf{T}'(t)$ instead, since $||\mathbf{T}|| = 1$, we only capture the change in direction. More formally we can see the result is normal to the curve by showing it's perpendicular to $\mathbf{T}(t)$:
   $$\begin{aligned}\mathbf{T}(t) \cdot \mathbf{T}(t) &= ||\mathbf{T}|| = 1 \\ \frac{d}{dt}\left(\mathbf{T}(t) \cdot \mathbf{T}(t)\right) &= 0 \\ 2\mathbf{T}'(t) \cdot T(t) &= 0\end{aligned}$$
   Since $\mathbf{T}' \perp \mathbf{T}$ and $\mathbf{N}$ is a multiple of $\mathbf{T}'$ we know $\mathbf{N} \perp \mathbf{T}$.

   Example: Find $\mathbf{T}$ and $\mathbf{N}$ at $(4, 2)$ on the curve $x = y^2$ by parametrizing as $\mathbf{r}(t) = t^2\,\mathbf{i} + t\,\mathbf{j}$ and working it out.

4. Tangential and normal components of acceleration:

   Acceleration breaks down into two components, one in the direction of motion and one perpendicular to it. These turn out to be multiples of $\mathbf{T}$ and $\mathbf{N}$ and in fact:
   $$\mathbf{a} = a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N}$$
   where:

   - The tangential component of acceleration is: $a_{\mathbf{T}} = \frac{\mathbf{v}\cdot\mathbf{a}}{||\mathbf{v}||}$
   - The normal component of acceleration is: $a_{\mathbf{N}} = \frac{||\mathbf{v}\times\mathbf{a}||}{||\mathbf{v}||}$

   Example: Find $a_{\mathbf{T}}$ and $a_{\mathbf{N}}$ at $t = 1$ for $\mathbf{r}(t) = 2t\,\mathbf{i} + t^2\,\mathbf{j} + 1/3t^3\,\mathbf{k}$.