Math 241 Section 13.6: The Gradient
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1. Definition
   The gradient of f is denoted either Grad$f$ or ∇f (note that ∇ is pronounced "nabla" and comes from the Hellenistic Greek word $\nu\alpha\beta\lambda\alpha$ for a Phoenician harp) and is defined by:

   $$\nabla f = f_x\,\mathbf{i} + f_y\,\mathbf{j} + f_z\,\mathbf{k}$$

   Note that this is a vector and dhe $+f_z\,\mathbf{k}$ only appears in the 3D case.
   Examples.

2. Basic properties

   (a) Observe that since $||\mathbf{u}|| = 1$ we have

   $$D_{\mathbf{u}} f = \mathbf{u} \cdot \nabla f = ||\mathbf{u}||||\nabla f|| \cos\theta = ||\nabla f|| \cos\theta$$

   Where θ is the angle between $\mathbf{u}$ and ∇f.
   It follows that $D_{\mathbf{u}} f$ is largest when $\theta = 0$ in which case $\mathbf{u}$ points in the same direction as ∇f and $D_{\mathbf{u}} f$ equals $||\nabla f||$.

   (b) First this means that ∇f points in the direction of maximum instantaneous increase of f.

   (c) Second this means that the largest possible $D_{\mathbf{u}} f$ is in fact $||\nabla f||$.

   (d) To put (b) and (c) together: Different $\mathbf{u}$ give different values for $D_{\mathbf{u}} f$. The largest value is when $\mathbf{u} = \nabla f$ and that largest value is $||\nabla f||$.

   Example. If the temp at (x, y) is $f(x, y) = x^2 y$ and a bug is at $(1, 2)$ in which direction does it detect the greatest increase in temperature and what is that increase?

3. Normal/Perpendicular properties

   (a) $\nabla f(x, y)$ is normal to the level curve of $f(x, y)$ at (x, y).
   Example: Find a vector $\perp$ to $y = x^2$ at $(3, 9)$.
   Solution: Set $f(x, y) = y - x^2$ then $\nabla f = -2x\,\mathbf{i} + 1\,\mathbf{j}$ and so $\nabla f(3, 9) = -18\,\mathbf{i} + 1\,\mathbf{j}$ works.

   (b) $\nabla f(x, y, z)$ is normal to the level surface of $f(x, y, z)$ at (x, y, z).
   Example.