Math 241 Section 13.7: Tangent Plane Approximation
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1. First recall how if $f(x)$ is a function and x_0 is an x-value we can draw the tangent line at $(x_0, f(x_0))$. Using point-slope form this line is $y - y_0 = f'(x_0)(x - x_0)$ or $y = y_0 + f'(x_0)(x - x_0)$. If x is close to x_0 then we have:

$$f(x) \approx y_0 + f'(x_0)(x - x_0)$$

While this is a weak approximation it's the beginning of the Taylor Series which is extremely useful:

$$f(x) = y_0 + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + ...$$

2. In the case of $f(x, y)$ we can do the same thing. We take the point (x_0, y_0) and construct the tangent plane. How do we do this? Well, the tangent plane passes through $(x_0, y_0, f(x_0, y_0))$ and has normal vector perpendicular to the graph of $f(x, y)$. Well, the graph of $z = f(x, y)$ is the level surface for the function of three variables $g(x, y, z) = f(x, y) - z$ and so we can obtain a perpendicular vector using the gradient and use this for **n**:

$$\begin{aligned} \nabla g(x, y, z) &= f_x\,\mathbf{i} + f_y\,\mathbf{j} - 1\,\mathbf{k} \\ \nabla g(x_0, y_0, f(x_0, y_0)) &= f_x(x_0, y_0)\,\mathbf{i} + f_y(x_0, y_0)\,\mathbf{j} - 1\,\mathbf{k} \\ \mathbf{n} &= f_x(x_0, y_0)\,\mathbf{i} + f_y(x_0, y_0)\,\mathbf{j} - 1\,\mathbf{k} \end{aligned}$$

Therefore the tangent plane has equation:

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - 1(z - f(x_0, y_0)) = 0$$

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

And similiarly now if (x, y) is close to (x_0, y_0) then:

$$f(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

(Note: The book rewrites this in a slightly different way which tends to be more confusing.)

This can then be used to approximate functions and is the first step in a 2D version of the Taylor Series (which we won't do).

Example: To approximate $\sqrt{3.02^2 + 6.95}$ we note this is close to $\sqrt{3^2 + 7}$ so we set $f(x, y) = \sqrt{x^2 + y}$ and use the approximation.

3. This can be expanded to 3D. For $f(x, y, z)$ if we have a point (x_0, y_0, z_0) then if (x, y, z) is close to (x_0, y_0, z_0) then

$$f(x, y, z) \approx f(x_0, y_0, z_0) + f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0)$$