Math 241 Section 14.2: Double Integrals in Polar
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1. Intro: Sometimes \( R \) can be easier to describe using polar. A region is described in polar as being between two angles \( \theta = \alpha \) and \( \theta = \beta \) and between two functions \( r = N(\theta) \) and \( r = F(\theta) \) where \( N \) is nearer to the origin and \( F \) is further from the origin.

2. Polar functions to rememeber:
   - Circles: \( r = a \), \( r = a \cos \theta \), \( r = a \sin \theta \)
   - Cardioids: \( r = a + a \cos \theta \), \( r = a + a \sin \theta \)
   - Converting rectangular: For example \( x = 2 \) becomes \( r \cos \theta = 2 \) or \( r = 2 \sec \theta \).

3. If \( R \) is described in polar then:
   \[
   \int \int_R f(x, y) \, dA = \int_\alpha^\beta \int_{N(\theta)}^{F(\theta)} f(r \cos \theta, r \sin \theta) \, r \, dr \, d\theta
   \]
   Note 1: the \( f(r \cos \theta, r \sin \theta) \) means convert \( f \) into polar.
   Note 2: Do not forget the additional \( r \). We’ll have a good explanation of this later.
   Example: \( \int \int_R x \, dA \) where \( R \) is the semicircle \( x^2 + y^2 \leq 9 \) with \( x \geq 0 \).
   Example: \( \int \int_R 1 \, dA \) where \( R \) is the region inside \( r = 1 + \cos \theta \) and outside \( r = 1 \).

4. Note about 14.1 and 14.2: Sometimes an iterated integral has been set up one way (VS, HS, P) and is easier another way. In this case we might rewrite it.
   Example: The integral
   \[
   \int_0^1 \int_{x\sqrt{3}}^{\sqrt{4-x^2}} \sqrt{x^2 + y^2} \, dy \, dx
   \]
   is particular icky. The region \( R \) however is simply the pie-slice in the first quadrant inside \( r = 2 \) and between \( \theta = \pi/3 \) and \( \theta = \pi/2 \). Therefore we can rewrite it as
   \[
   \int_{\pi/3}^{\pi/2} \int_0^2 r^2 \, dr \, d\theta
   \]
   which is much more manageable.