Math 241 Section 14.5: Triple Integrals in Polar
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1. Introduction: Cylindrical is like polar plus z however many surfaces can look strange in cylindrical. Examples:

   (a) $z = x^2 + y^2$ becomes $z = r^2$.

   (b) $z = \sqrt{x^2 + y^2}$ becomes $z = r$.

   (c) $x^2 + y^2 + z^2 = 9$ becomes $r^2 + z^2 = 9$ or the top half $z = \sqrt{9 - r^2}$.

   (d) $z = 2 - x - y$ becomes $z = 2 - r\cos\theta - r\sin\theta$.

   (e) $r = \sin\theta$ becomes a cylinder, as does $r = \cos\theta$ and $r = 3$.

2. The method: If R is parametrized in polar then:

$$\int\int\int_D f(x,y,z)dV = \int_\alpha^\beta \int_{N(\theta)}^{F(\theta)} \int_{\text{Floor}(r,\theta)}^{\text{Ceiling}(r,\theta)} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$$

   As with triple integrals in rectangular the first two integrals take care of R. The top and bottom functions must be rewritten in terms of r and θ and the integrand must be rewritten too.

   Example: The mass of the ice-cream cone inside $z = \sqrt{(3x^2 + 3y^2)}$ and inside $x^2 + y^2 + z^2 = 4$. It's often tricky to identify R and even the top and bottom functions are often confusing. Here I used $f(x,y,z) = z^2$ for the density.

   Example: The volume of the solid inside $r = \sin(\theta)$, below $z = 9 - x^2 - y^2$ and above the xy-plane