Math 241 Section 14.5: Triple Integrals in Polar
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1. Introduction: Cylindrical is like polar plus \( z \) however many surfaces can look strange in cylindrical.

Examples:
(a) \( z = x^2 + y^2 \) becomes \( z = r^2 \).
(b) \( z = \sqrt{x^2 + y^2} \) becomes \( z = r \).
(c) \( x^2 + y^2 + z^2 = 9 \) becomes \( r^2 + z^2 = 9 \) or the top half \( z = \sqrt{9 - r^2} \).
(d) \( z = 2 - x - y \) becomes \( z = 2 - r \cos \theta - r \sin \theta \).
(e) \( r = \sin \theta \) becomes a cylinder, as does \( r = \cos \theta \) and \( r = 3 \).

2. The method: If \( R \) is parametrized in polar then:

\[
\int \int \int_D f(x,y,z)\,dV = \int_\alpha^\beta \int_{\text{Floor}(r,\theta)}^{\text{Ceiling}(r,\theta)} \int_{N(\theta)}^{F(\theta)} f(r \cos \theta, r \sin \theta, z) \,r \,dz \,dr \,d\theta
\]

As with triple integrals in rectangular the first two integrals take care of \( R \). The top and bottom functions must be rewritten in terms of \( r \) and \( \theta \) and the integrand must be rewritten too.

Example: The mass of the ice-cream cone inside \( z = \sqrt{(3x^2 + 3y^2)} \) and inside \( x^2 + y^2 + z^2 = 4 \). It’s often tricky to identify \( R \) and even the top and bottom functions are often confusing. Here I used \( f(x, y, z) = z^2 \) for the density.

Example: The volume of the solid inside \( r = \sin(\theta) \), below \( z = 9 - x^2 - y^2 \) and above the \( xy \)-plane