Math 241 Section 14.6: Triple Integrals in Spherical
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1. Introduction: Introduce spherical coordinates. We locate a point using three variables:

   θ is the familiar one, the angle from the positive x-axis.
   ϕ is the angle measured from the positive z-axis downwards, so $0 \leq \phi \leq \pi$.
   ρ is the distance from the origin.

   Graph some points (very few) and then show a picture which illustrates how to change these to x, y, z which we'll need for substitution:

$$\begin{aligned} x &= \rho \sin\phi \cos\theta \\ y &= \rho \sin\phi \sin\theta \\ z &= \rho \cos\phi \\ x^2 + y^2 + z^2 &= \rho^2 \end{aligned}$$

2. Classic examples of equations in spherical:
   (a) Sphere $\rho = 3$
   (b) Cone $\phi = \pi/4$
   (c) The plane $z = 2$ becomes $\rho\cos\phi = 2$ or $\rho = 2\sec\phi$.

3. Describing objects in spherical: A solid D will be described as having:

$$\begin{gathered} \alpha \leq \theta \leq \beta \\ \gamma \leq \phi \leq \delta \\ near(\theta, \phi) \leq \rho \leq far(\theta, \phi) \end{gathered}$$

4. Integration: If D is described in spherical then:

$$\int\int\int_D f(x, y, z)\, dV = \int_\alpha^\beta \int_\gamma^\delta \int_{near(\theta,\phi)}^{far(\theta,\phi)} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

   NOTE 1: The $\rho^2 \sin\phi$ is like the polar/cylindrical r. We just have to remember for now to put it there.

   NOTE 2: The book allows the middle iterated integral to have limits which are functions of theta - see the $h_1(\theta)$ and $h_2(\theta)$ in Thm 14.11 on p.947. In practice these are always constants so I just gave those limits as γ and δ.

   Example: The solid between the spheres $\rho = 1$ and $\rho = 2$.

   Example: If we just take the top half of the previous.

   Example: If we just take the part of the previous having $x \leq 0$, $y \geq 0$ and $z \geq 0$. This one is nice to draw.

   Example: The volume of the ice-cream cone inside $\phi = \pi/6$ and and inside $\rho = 5$.

   Example: If we chop off the bottom at $z = 1$.