Math 241 Section 14.8: Change of Variables
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1. Intro Part I: For double integrals, sometimes VS, HS and polar are insufficient for parallelograms, ellipses and other quirky shapes.

2. Intro Part II: Historical note - consider the integral $\int_0^1 \sqrt{1-x^2}\,dx$. We put $x = \sin u$ and get $\int_0^{\pi/2} \sqrt{1-\sin^2 u}\cos u\,du$ and so three things have changed:

   The interval changes, the integrand changes, and the dx is replaced by $\cos u du$.

3. A change of vars is basically a 2D version of this. Our goal is to do this for a double integral. The change of variables formula works as follows: If R is a not-so-nice region and we want to evaluate $\int\int_R f(x,y)\,dA$ and if we can do a substitution $x = g(u,v)$ and $y = h(u,v)$ as functions of u, v which changes R (in the xy-plane) into S (in the uv-plane) then:

$$\int\int_R f(x,y)\,dA = \int\int_S f(g(u,v),h(u,v))\,|J(x,y)|\,dA \text{ where } J(x,y) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

   Notes:

   - Often the change of variables is given by $u =$ and $v =$. In this case we need to solve for $x =$ and $y =$ only if we need them for the integrand.
   - We can often use: $J(x,y) = 1 \div J(u,v)$ where $J(u,v) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$

4. Three classic examples.

   Example: Find $\int\int_R x dA$ where R is the parallelogram bounded by $y = 1-x$, $y = 3-x$, $y = \frac{1}{2}x+1$, and $y = \frac{1}{2}x + 4$. We rewrite the edges as $x + y = 1$, $x + y = 3$, $y - \frac{1}{2}x = 1$, and $y - \frac{1}{2}x = 4$ and substitute $u = x + y$ and $v = y - \frac{1}{2}x$. Then S is a rectangle. Since the integrand is just x we need to solve for x. We might as well find y too and then find $J(x,y)$ directly.
   Note: If the integrand had been something like $x + y$ then we know it becomes u and we could have used the alternate method for $J(x,y)$.

   Example: Find $\int\int_R y^2 dA$ where R is the ellipse $x^2/9+y^2/4 = 1$. We rewrite as $(x/3)^2+(y/2)^2 = 1$ and substitute $u = x/3$ and $v = y/2$. Then S is a circle and then we go to polar.

   Example: Find $\int\int_R \frac{x}{y} dA$ where R is the region bounded by $y = x$, $y = 2x$, $y = \frac{1}{x}$, and $y = \frac{2}{x}$. We rewrite the edges as $\frac{y}{x} = 1$, $\frac{y}{x} = 3$, $xy = 1$ and $xy = 2$ and then substitute $u = \frac{y}{x}$ and $v = xy$. Then S is a rectangle. Notice here that the integrand is y/x which is u so we never need to solve for x and y directly so we can use the alternate method for $J(x,y)$.