Math 241 Section 15.1: Vector Fields
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1. Definition: A *vector field* is a function which assigns a vector to each point. This can either be 2D:

   Example: $\mathbf{F}(x, y) = xy\,\mathbf{i} + 2x^2\,\mathbf{j}$

   Or it can be 3D:

   Example: $\mathbf{F}(x, y, z) = ye^z\,\mathbf{i} + xz\,\mathbf{j} + y^2\,\mathbf{k}$

   We can sketch a VF by simply plotting a vector at each point for a nice selection of points. Example: $\mathbf{F}(x, y) = \frac{1}{5}y\,\mathbf{i} - \frac{1}{5}x\,\mathbf{j}$ is good to draw.

   Vector fields can be thought of as representing any sort of force field, fluid flow, etc.

2. Associated definitions. For a vector field $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ define:

   (a) Divergence: $\nabla \cdot \mathbf{F} = M_x + N_y + P_z$. This is a scalar and measures net gain/loss of fluid at a point. A source is a point with positive divergence and a sink is a point with negative divergence.
   Example: Anything.

   (b) Curl: $\nabla \times \mathbf{F}$ which is defined by taking the cross-product $\left(\frac{\partial}{\partial x}\,\mathbf{i} + \frac{\partial}{\partial y}\,\mathbf{j} + \frac{\partial}{\partial z}\,\mathbf{k}\right) \times \mathbf{F}$ and doing the corresponding partials. Loosely speaking the curl measures the axis of rotation of the fluid at a point.
   Example: Anything.

3. Conservative vector fields.

   (a) We've seen VF before. Recall the gradient $\nabla f = f_x\,\mathbf{i} + f_y\,\mathbf{j} + f_z\,\mathbf{k}$. This gives us a VF.
   Example: If $f(x, y, z) = xy^2z$ then $\nabla f = y^2z\,\mathbf{i} + 2xyz\,\mathbf{j} + xy^2\,\mathbf{k}$.
   Definition: A vector field $\mathbf{F}$ is conservative if there is some f so that $\mathbf{F} = \nabla f$. This f is called a potential function.
   Example: $\mathbf{F}(x, y, z) = y^2z\,\mathbf{i} + 2xyz\,\mathbf{j} + xy^2\,\mathbf{k}$ is conservative.

   (b) Theorems:

   - If $\mathbf{F}$ is conservative then $\nabla \times \mathbf{F} = \mathbf{0}$.
   - If $\nabla \times \mathbf{F} \neq \mathbf{0}$ then $\mathbf{F}$ is not conservative.
     Example: $\mathbf{F}(x, y, z) = xy\,\mathbf{i} + y\,\mathbf{j} + xz\,\mathbf{k}$ is not because $\nabla \times \mathbf{F} \neq \mathbf{0}$.
   - If $\nabla \times \mathbf{F} = \mathbf{0}$ and if $\mathbf{F}$ is defined at all (x, y, z) then $\mathbf{F}$ is conservative.

   (c) Finding a potential function: If $\mathbf{F}$ is conservative then it will be useful to find a corresponding potential function f. This can often be guessed but it's useful to have a process. Here's an example with a process:
   Example: $\mathbf{F}(x, y, z) = 2xy\,\mathbf{i} + \left[x^2 + z\right]\,\mathbf{j} + [y + 2z]\,\mathbf{k}$
   Process: We want $f(x, y, z)$ with $f_x(x, y, z) = 2xy$, $f_y(x, y, z) = x^2 + z$ and $f_z(x, y, z) = y + 2z$.
   Step 1: Since $f_x(x, y, z) = 2xy$ we have $f(x, y, z) = x^2y + g(y, z)$.
   Step 2: From here $f_y(x, y, z) = x^2 + g_y(y, z)$ but this must equal $x^2 + z$ so $x^2 + g_y(y, z) = x^2 + z$ so $g_y(y, z) = z$ so $g(y, z) = yz + h(z)$ and so $f(x, y, x) = x^2y + yz + h(z)$.
   Step 3: From here $f_z(x, y, z) = y + h'(z)$ but this must equal $y + 2z$ so $y + h'(z) = y + 2z$ so $h'(z) = 2z$ so $h(z) = z^2 + C$ and so $f(x, y, z) = x^2y + yz + z^2 + C$.
   Since C can be any constant we typically use $C = 0$.