Math 241 Section 15.2: Line Integrals of Functions and VFs
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1. Line Integrals of Functions:

   (a) Situation: If C is a curve representing a wire and if $f(x, y, z)$ is the density at (x, y, z) then we can ask the mass. This is the line integral of $f(x, y, z)$ over C denoted $\int_C f(x, y, z)\, ds$.

   (b) Evaluation (Only Way): We parametrize C as $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$ for $a \leq t \leq b$ and then

   $$\int_C f(x, y, z)\, ds = \int_a^b f(x(t), y(t), z(t))||\mathbf{r}'(t)||\, dt$$

   Example: C is the semicircle $x^2 + y^2 = 4$ with $y \geq 0$ and density $f(x, y) = y$.

2. Line Integrals of Vector Fields:

   (a) Situation: If C is a curve representing the path of an object through a vector field (force field) $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ we can ask the work done by F on the object. This is the line integral of $\mathbf{F}(x, y, z)$ over C denoted either $\int_C \mathbf{F} \cdot d\mathbf{r}$ or $\int_C M\, dx + N\, dy + P\, dz$.

   (b) Evaluation (Way 1 of 4): We parametrize C as $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$ for $a \leq t \leq b$ and then

   $$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(x(t), y(t), z(t)) \cdot \mathbf{r}'(t)\, dt$$

   Example:
   Example:

   (c) Note 1: Direction of C matters, changing the direction negates the integral, and why.

   (d) Note 2: Notation can be confusing. Observe the difference between $\int_C x\, dx$, $\int_C x\, ds$, $\int_a^b x\, dx$.