Math 241 Section 15.5: Surface Integrals of Functions  
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1. Intro: Suppose $\Sigma$ is a surface and $f(x, y, z)$ is defined at each point in $\Sigma$. If we chop $\Sigma$ into little rectangles and take the area of each multiplied by $f$ at some point in the rectangle, then let the size of the rectangles to go zero, the result is $\int \int_{\Sigma} f \, dS$. Some uses:

- If $f(x, y, z) = 1$ we get the surface area of $\Sigma$.
- If $f(x, y, z)$ is the density (mass or electrical charge or whatever) at any point is $f(x, y, z)$ then we get the total (mass, charge, whatever).

2. Method of evaluation; Only one way:
   Parametrize $\Sigma$ as $\mathbf{r}(u, v) = x(u, v) \mathbf{i} + y(u, v) \mathbf{j} + z(u, v) \mathbf{k}$ for $u, v$ in the region $R$ in the $uv$-plane ($R$ is usually described simply by inequalities on $u$ and $v$) and then:

$$\int \int_{\Sigma} f(x, y, z) \, dS = \int \int_{R} f(x(u, v), y(u, v), z(u, v)) ||\mathbf{r}_u \times \mathbf{r}_v|| \, dA$$

Note: We’re used to using $R$ to denote a region in the $xy$-plane but that’s not really the case here. Instead $R$ is a region in the $uv$-plane for whatever variables we’re using, but it’s easier to think of $R$ as a set of $(u, v)$ usually described by a pair of inequalities.

3. Examples
   
   Example: $f(x, y, z) = xyz$ for $\Sigma$ the part of $z = 9 - x^2 - y^2$ above the rectangle with $0 \leq x \leq 1$ and $0 \leq y \leq 2$. Here we use $\mathbf{r}(x, y)$.

   Example: $f(x, y, z) = xyz$ for $\Sigma$ the part of the cylinder $x^2 + y^2 = 4$ between $z = 1, 5$. Here we use $\mathbf{r}(z, \theta)$.

   Example: $f(x, y, z) = x^2 z$ for $\Sigma$ the part of $z = 7 - x$ inside $r = 2 \sin(\theta)$. Here we use $\mathbf{r}(r, \theta)$. This is often a bit confusing because we have $0 \leq \theta \leq \pi$ (generally okay) and $0 \leq r \leq 2 \sin \theta$ (often confusing).

   NOTE: An important point - I really want to be sure that this is very step-by-step, meaning we go from an integral over $\Sigma$ to an integral over $R$ to an iterated integral and that we don’t skip the middle step.