Math 241 Section 15.5: Surface Integrals of Functions
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1. Intro: Suppose Σ is a surface and $f(x,y,z)$ is defined at each point in Σ. If we chop Σ into little rectangles and take the area of each multiplied by f at some point in the rectangle, then let the size of the rectangles to go zero, the result is $\int\int_\Sigma f\,dS$. Some uses:

   - If $f(x,y,z) = 1$ we get the surface area of Σ.
   - If $f(x,y,z)$ is the density (mass or electrical charge or whatever) at any point is $f(x,y,z)$ then we get the total (mass,charge, whatever).

2. Method of evaluation; Only one way:

   Parametrize Σ as $\mathbf{r}(u,v) = x(u,v)\,\mathbf{i} + y(u,v)\,\mathbf{j} + z(u,v)\,\mathbf{k}$ for u, v in the region R in the uv-plane (R is usually described simply by inequalities on u and v) and then:

   $$\int\int_\Sigma f(x,y,z)\,dS = \int\int_R f(x(u,v), y(u,v), z(u,v))||\mathbf{r}_u \times \mathbf{r}_v||\,dA$$

   Note: We're used to using R to denote a region in the xy-plane but that's not really the case here. Instead R is a region in the uv-plane for whatever variables we're using, but it's easier to think of R as a set of (u,v) usually described by a pair of inequalities.

3. Examples

   Example: $f(x,y,z) = xyz$ for Σ the part of $z = 9 - x^2 - y^2$ above the rectangle with $0 \leq x \leq 1$ and $0 \leq y \leq 2$. Here we use $\mathbf{r}(x,y)$.

   Example: $f(x,y,z) = xyz$ for Σ the part of the cylinder $x^2 + y^2 = 4$ between $z = 1, 5$. Here we use $\mathbf{r}(z,\theta)$.

   Example: $f(x,y,z) = x^2 z$ for Σ the part of $z = 7 - x$ inside $r = 2\sin(\theta)$. Here we use $\mathbf{r}(r,\theta)$. This is often a bit confusing because we have $0 \leq \theta \leq \pi$ (generally okay) and $0 \leq r \leq 2\sin\theta$ (often confusing).

   NOTE: An important point - I really want to be sure that this is very step-by-step, meaning we go from an integral over Σ to an integral over R to an iterated integral and that we don't skip the middle step.