Math 241 Section 15.6: Surface Integrals of Vector Fields
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1. An oriented surface is a surface with a chosen direction through the surface. More specifically it's a continuous choice of unit normal vectors at each point on the surface.

2. If $\Sigma$ is an oriented surface immersed in a fluid with fluid flow $\mathbf{F}$ then the rate at which $\mathbf{F}$ flows through $\Sigma$ in the direction of the orientation is given by the surface integral of the vector field, denoted

$$\int \int_{\Sigma} \mathbf{F} \cdot \mathbf{n} \, dS$$

3. The most basic method of evaluation: Parametrize $\Sigma$ using $\mathbf{r}(u, v) = x(u, v) \mathbf{i} + y(u, v) \mathbf{j} + z(u, v) \mathbf{k}$ with $u, v$ constrained by inequalities described by $R$. Then:

$$\int \int_{\Sigma} \mathbf{F} \cdot \mathbf{n} \, dS = \pm \int \int_{R} \mathbf{F}(x(u, v), y(u, v), z(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) \, dA$$

In which the $\pm$ is determined as follows:

- Use $+$ if the vectors $\mathbf{r}_u \times \mathbf{r}_v$ match the direction of the orientation of $\Sigma$.
- Use $-$ if they are opposite.

4. Examples:

Example: $\Sigma$ is the part of $x + 2y + 2z = 6$ in the first octant oriented downwards and with $\mathbf{F}(x, y, z) = xy \mathbf{i} + y \mathbf{j} + xz \mathbf{k}$.

Example: $\Sigma$ is the part of $x^2 + y^2 = 9$ with $0 \leq z \leq 5$ oriented outwards and with $\mathbf{F}(x, y, z) = z \mathbf{i} + xy \mathbf{j} + y^2 \mathbf{k}$.