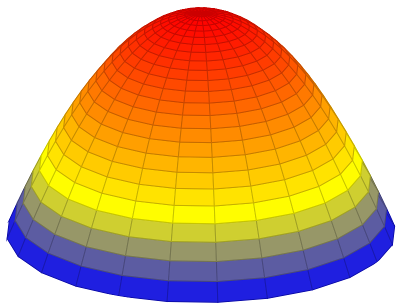
Math 241 Section 15.6: Surface Integrals of Vector Fields
Dr. Justin O. Wyss-Gallifent

1. An oriented surface is a surface with a chosen direction through the surface. More specifically it's a continuous choice of unit normal vectors at each point on the surface.



2. If Σ is an oriented surface immersed in a fluid with fluid flow $\mathbf{F}$ then the rate at which $\mathbf{F}$ flows through Σ in the direction of the orientation is given by the surface integral of the vector field, denoted
$$\int\int_{\Sigma} \mathbf{F}\cdot\mathbf{n}\,dS$$

3. The most basic method of evaluation: Parametrize Σ using $\mathbf{r}(u,v) = x(u,v)\,\mathbf{i} + y(u,v)\,\mathbf{j} + z(u,v)\,\mathbf{k}$ with u, v constrained by inequalities described by R. Then:
$$\int\int_{\Sigma} \mathbf{F}\cdot\mathbf{n}\,dS = \pm\int\int_{R} \mathbf{F}(x(u,v), y(u,v), z(u,v))\cdot(\mathbf{r}_u\times\mathbf{r}_v)\,dA$$
In which the $\pm$ is determined as follows:
   - Use $+$ if the vectors $\mathbf{r}_u\times\mathbf{r}_v$ match the direction of the orientation of Σ.
   - Use $-$ if they are opposite.

4. Examples:

   Example: Σ is the part of $x + 2y + 2z = 6$ in the first octant oriented downwards and with $\mathbf{F}(x,y,z) = xy\,\mathbf{i} + y\,\mathbf{j} + xz\,\mathbf{k}$.

   Example: Σ is the part of $x^2 + y^2 = 9$ with $0 \le z \le 5$ oriented outwards and with $\mathbf{F}(x,y,z) = z\,\mathbf{i} + xy\,\mathbf{j} + y^2\,\mathbf{k}$.