Math 241 Section 15.8: The Divergence Theorem (Gauss' Theorem)
Dr. Justin O. Wyss-Gallifent

1. The Divergence Theorem: If Σ is a surface which completely surrounds a solid D and if Σ has outwards orientation then:

$$\int\int_{\Sigma} \mathbf{F} \cdot \mathbf{n}\, dS = \int\int\int_{D} \nabla \cdot \mathbf{F}\, dV$$

   Two immediate notes:

   - Σ is the surface but D is the solid inside it.
   - How we do $\int\int\int_D \nabla \cdot \mathbf{F}\, dV$ depends on D. Could be rectangular, cylindrical or spherical.

   Example: Suppose Σ is the cylinder $x^2 + y^2 = 4$ between $z = 0$ and $z = 5$ with the caps at the ends, oriented outwards, evaluate $\int\int_{\Sigma}(x^2\,\mathbf{i} + xz\,\mathbf{j} + z\,\mathbf{k}) \cdot \mathbf{n}\, dS$.

2. Notes

   (a) If Σ is oriented inwards we can negate.

   (b) If $\nabla \cdot \mathbf{F}$ is a constant then the result is that constant times the volume of D. This is only useful if the volume can be conveniently calculated.

   Example of (a) and (b): Suppose Σ is the top hemisphere part of $x^2 + y^2 + z^2 = 9$ along with the base, oriented inwards, evaluate $\int\int_{\Sigma}(2x\,\mathbf{i} + 5y\,\mathbf{j} + 7z\,\mathbf{k}) \cdot \mathbf{n}\, dS$.

   (c) Σ must completely surround D to do this. In that sense it's like a 3D version of Green's Theorem.

   Example of (c): In the previous problem if we hadn't included the base then the Divergence Theorem would not apply.

   (d) This Theorem makes sense. The surface integral is measuring the fluid flow across the surface while the triple integral is measuring the "total divergence" which can be thought of as the "total sink/source" nature of the vector field inside the object. It makes sense that these are equal because the fluid can only appear (source) or disappear (sink) equal to how it crosses the boundary.