1. A Bit of History and Introduction: Suppose \( H(x, y) \) is a function and \( y \) is a function of \( x \). Then by the chain rule we know \( \frac{d}{dx} H(x, y) = H_x(x, y) + H_y(x, y) \frac{dy}{dx} \).

So now consider the following differential equation:

\[
3x^2y^2 + 2x^3y \frac{dy}{dx} = 0
\]

You may notice that the left side looks like the result of the chain rule and is actually so, when \( H(x, y) = x^3y^2 \). Don’t worry about if there’s a formal method for where \( H(x, y) \) comes from for now, just notice that \( H_x(x, y) = 3x^2y^2 \) and \( H_y(x, y) = 2x^3y \). What this means is that the differential equation may be rewritten by undoing the chain rule on the left:

\[
3x^2y^2 + 2x^3y \frac{dy}{dx} = 0
\]

So then when the derivative of something is zero, that thing is a constant:

\[
\frac{d}{dx} [x^3y^2] = 0
\]

\[
x^3y^2 = C
\]

and we’ve solved it, at least implicitly!

2. Definition and Method: A differential equation is *exact* if it has the form:

\[
H_x(x, y) + H_y(x, y) \frac{dy}{dx} = 0
\]

for some function \( H(x, y) \). When a differential equation is exact, solving implicitly is as easy as finding \( H(x, y) \) and setting \( H(x, y) = C \) for any constant.

Here are a few exact differential equations. For each, \( H(x, y) \) is written in the middle and the implicit solution to the right.

<table>
<thead>
<tr>
<th>Exact DE</th>
<th>( H(x, y) )</th>
<th>Solution to DE</th>
</tr>
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<tbody>
<tr>
<td>( y + x \frac{dy}{dx} = 0 )</td>
<td>( H(x, y) = xy )</td>
<td>( xy = C )</td>
</tr>
<tr>
<td>( y + (x + 2y) \frac{dy}{dx} = 0 )</td>
<td>( H(x, y) = xy + y^2 )</td>
<td>( xy + y^2 = C )</td>
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<tr>
<td>( \frac{1}{y} - x \frac{dy}{dx} = 0 )</td>
<td>( H(x, y) = \frac{x}{y} )</td>
<td>( \frac{x}{y} = C )</td>
</tr>
<tr>
<td>( y \cos(xy) + x \cos(xy) \frac{dy}{dx} = 0 )</td>
<td>( H(x, y) = \sin(xy) )</td>
<td>( \sin(xy) = C )</td>
</tr>
</tbody>
</table>
3. **Detecting Exactness and Finding H:** There is a trick to detecting whether a differential equation is exact. If the differential equation has the form:

\[ M + N \frac{dy}{dx} = 0 \]

then it is exact if and only if \( M_y = N_x \). You can test all the ones above. Then you can check that this next one is not exact:

\[ xy + y \frac{dy}{dx} = 0 \]

In this case \( M_y = x \) and \( N_x = 0 \). Not equal, not exact.

Once you know that your differential equation is exact, often you can guess at \( H(x, y) \). However if you’re struggling, there’s a systematic method for finding it. Here’s an example from above:

\[ y + (x + 2y) \frac{dy}{dx} = 0 \]

We want \( H(x, y) \) with (A) \( H_x(x, y) = y \) and (B) \( H_y(x, y) = x + 2y \). Observe:

We want (A):

\[ H_x(x, y) = y \]

This tells us that:

\[ H(x, y) = xy + h(y) \]

From this line:

\[ H_y(x, y) = x + h'(y) \]

But from (B):

\[ H_y(x, y) = x + 2y \]

Set these equal:

\[ x + h'(y) = x + 2y \]

Solve for \( h'(y) \):

\[ h'(y) = 2y \]

Find \( h(y) \):

\[ h(y) = y^2 + D \]

Put back into second line:

\[ H(x, y) = xy + y^2 + D \]

We can choose any \( D \) so choose \( D = 0 \) to get \( H(x, y) = xy + y^2 \).

**Example:** Find \( H(x, y) \) to solve \( x + 1 + \frac{1}{y} = \frac{xy}{y^2} \frac{dy}{dx} = 0 \). Follow the exact procedure above, here we want (A) \( H_x(x, y) = x + 1 + \frac{1}{y} \) and (B) \( H_y(x, y) = -\frac{x}{y^2} \):

We want (A):

\[ H_x(x, y) = x + 1 + \frac{1}{y} \]

This tells us that:

\[ H(x, y) = \frac{x}{2} y^2 + x + \frac{y}{y} + h(y) \]

From this line:

\[ H_y(x, y) = -\frac{x}{y^2} + h'(y) \]

But from (B):

\[ H_y(x, y) = -\frac{x}{y^2} \]

Set these equal:

\[ -\frac{x}{y^2} + h'(y) = -\frac{x}{y^2} \]

Solve for \( h'(y) \):

\[ h'(y) = 0 \]

Find \( h(y) \):

\[ h(y) = D \]

Put back into second line:

\[ H(x, y) = \frac{1}{2} x^2 + x + \frac{x}{y} + D \]

Then choose \( D = 0 \) to get \( H(x, y) = \frac{1}{2} x^2 + x + \frac{x}{y} \) and the solution to our DE is \( \frac{1}{2} x^2 + x + \frac{x}{y} = C \).
4. **Almost Exact:** It’s not uncommon to have a differential equation which is not quite exact but can be made exact by multiplying through by some function called an *integrating factor*. For example the differential equation

\[ 2y + x \frac{dy}{dx} = 0 \]

is not exact because \( M_y = 2 \) and \( N_x = 1 \) so \( M_y \neq N_x \). But if we multiply through by \( x \) we get the new differential equation

\[ 2xy + x^2 \frac{dy}{dx} = 0 \]

which is exact because \( M_y = 2x \) and \( N_x = 2x \). Now \( H(x, y) = x^2 y \) and the solution is \( x^2 y = C \).

The question is how to come up with this integrating factor. This can be challenging but we’ll look at two simple cases. The key is that we’ve got our non-exact differential equation

\[ M + N \frac{dy}{dx} = 0 \]

and we wish to multiply through by some \( \mu(x, y) \) such that the new differential equation

\[ M \mu + N \mu \frac{dy}{dx} = 0 \]

is exact. To be exact we’d need

\[ (M \mu)_y = (N \mu)_x \]

\[ M_y \mu + M \mu_y = N_x \mu + N \mu_x \]

While this seems tricky (it’s actually a partial differential equation!) we will only encounter the special cases when \( \mu \) is a function of just \( x \) or just \( y \).

The key is to take the above equation and say:

- If \( \mu \) is a function of only \( x \) then \( \mu_y = 0 \). Rewriting this equation, can we see a \( \mu(x) \) which would make this equation true?
- If \( \mu \) is a function of only \( y \) then \( \mu_x = 0 \). Rewriting this equation, can we see a \( \mu(y) \) which would make this equation true?

Note: We will only look at examples where \( \mu \) is either a function of only \( x \) or only \( y \) and where \( \mu \) is easy to figure out visually. Going beyond this can get seriously difficult.
5. Examples:

**Example 1:** Consider the non-exact differential equation we’ve seen before:

\[ 2y + x \frac{dy}{dx} = 0 \]

Here \( M = 2y \) and \( N = x \). We’d like:

\[
M_y \mu + M \mu_y = N_x \mu + N \mu_x \\
2\mu + 2y\mu_y = 1\mu + x\mu_x
\]

If \( \mu = \mu(x) \) then \( \mu_y = 0 \) and this becomes:

\[
2\mu = \mu + x\mu_x \\
x\mu_x = \mu \\
\mu_x = \frac{\mu}{x}
\]

We can see that \( \mu(x) = x \) does the job. This is then our integrating factor and we multiply our original differential equation through by it to get the exact differential equation:

\[ 2xy + x^2 \frac{dy}{dx} = 0 \]

which has \( H(x, y) = x^2y \) and solution \( x^2y = C \).

**Example 2:** Consider the non-exact differential equation

\[ y + (x + xy) \frac{dy}{dx} = 0 \]

Here \( M = y \) and \( N = x + xy \). We’d like:

\[
M_y \mu + M \mu_y = N_x \mu + N \mu_x \\
1\mu + y\mu_y = (1 + y)\mu + (x + xy)\mu_x
\]

If \( \mu = \mu(y) \) then \( \mu_x = 0 \) and this becomes:

\[
1\mu + y\mu_y = (1 + y)\mu \\
\mu + y\mu_y = \mu + y\mu \\
\mu_y = \mu
\]

We can see that \( \mu(y) = e^y \) does the job. This is then our integrating factor and we multiply our original differential equation through by it to get the exact differential equation:

\[ ye^y + (xe^y + xye^y) \frac{dy}{dx} = 0 \]

This has \( H(x, y) = xye^y \) and solution \( xye^y = C \).