1. Introduction:
   It’s far easier to manage systems of differential equations when we can rephrase them in the
   language of matrices and vectors. To that end, here are the essentials.

2. Basic Definitions:
   (a) A matrix is a rectangular array of numbers. It has size \( m \times n \) if there are \( m \) rows and \( n \)
   columns. Matrices are typically denoted using capital letters:

   **Example:** Here is a \( 3 \times 4 \) matrix:

   \[
   A = \begin{bmatrix}
   1 & 3 & -1 & 0 \\
   0 & 5 & 0 & 17 \\
   2 & 2 & -7 & 3 \\
   \end{bmatrix}
   \]

   (b) Most of the matrices in this class will be square, meaning they have the same number of rows
   and columns. Mostly we’ll deal with \( 2 \times 2 \) and \( 3 \times 3 \) matrices.

   (c) The identity matrix \( I_n \) is the square \( n \times n \) matrix with 1s on the main diagonal (upper-left
   to lower right) and 0’s elsewhere. When the size is clear from context we just write \( I \).

   **Example:**

   \[
   I_3 = \begin{bmatrix}
   1 & 0 & 0 \\
   0 & 1 & 0 \\
   0 & 0 & 1 \\
   \end{bmatrix}
   \]

   (d) The zero matrix is matrix of all zeros.

   (e) A vector is a matrix which is a single column. Vectors are usually denoted in lower-case with
   a bar over the letter.

   **Example:** \( \vec{a} = \begin{bmatrix}
   1 \\
   -3 \\
   2 \\
   \end{bmatrix} \)
3. Combining Matrices and Vectors:

(a) We can add matrices and vectors by adding matching entries provided they both have the same size.

**Example:** For example:
\[
\begin{bmatrix}
1 & 2 \\
3 & 4
\end{bmatrix}
+ 
\begin{bmatrix}
-1 & 0 \\
6 & 2
\end{bmatrix}
= 
\begin{bmatrix}
1-1 & 2+0 \\
3+6 & 4+2
\end{bmatrix}
= 
\begin{bmatrix}
0 & 2 \\
9 & 8
\end{bmatrix}
\]

(b) We can multiple an \(n \times n\) matrix \(A\) by a vector \(\vec{x}\) with \(n\) entries to get a new vector with \(n\) entries. We do this by multiplying each row of the matrix by the vector (element by element and add). This is easier to see:

**Example:** We have:
\[
\begin{bmatrix}
1 & 2 & -3 \\
0 & 4 & 7 \\
8 & -1 & 5
\end{bmatrix}
\begin{bmatrix}
5 \\
3 \\
2
\end{bmatrix}
= 
\begin{bmatrix}
(1)(5) + (2)(3) + (-3)(2) \\
(0)(5) + (4)(3) + (7)(2) \\
(8)(5) + (-1)(3) + (5)(2)
\end{bmatrix}
= 
\begin{bmatrix}
5 \\
26 \\
47
\end{bmatrix}
\]

(c) We can multiple an \(n \times n\) matrix by another \(n \times n\) matrix by multiplying the first matrix by each of the columns in the second matrix as if it were just a vector, then taking these new vectors an putting them together in a new matrix.

**Example:** Here it is with lots of brackets to help you see what’s going on:
\[
\begin{bmatrix}
2 & 1 \\
4 & 3
\end{bmatrix}
\begin{bmatrix}
-1 & 3 \\
5 & 9
\end{bmatrix}
= 
\begin{bmatrix}
2 & 1 & -1 & 3 \\
4 & 3 & 5 & 9
\end{bmatrix}
\begin{bmatrix}
2 & 1 \\
4 & 3
\end{bmatrix}
\begin{bmatrix}
-1 \\
5
\end{bmatrix}
= 
\begin{bmatrix}
(2)(-1) + (1)(5) \\
(4)(-1) + (3)(5)
\end{bmatrix}
\begin{bmatrix}
(2)(3) + (1)(9) \\
(4)(3) + (3)(9)
\end{bmatrix}
= 
\begin{bmatrix}
3 & 15 \\
9 & 39
\end{bmatrix}
\]

(d) It’s almost always the case that for matrices \(A\) and \(B\) that \(AB \neq BA\).

(e) The identity matrix acts like the number 1 in that for any matrix \(A\) we have:
\[
AI = IA = A
\]
4. Inverses:

(a) The **inverse** of an $n \times n$ matrix $A$ is another matrix denoted $A^{-1}$ such that $AA^{-1} = A^{-1}A = I$.

It’s like a “reciprocal” for matrices.

(b) For the $2 \times 2$ size there is a formula:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

**Example:** For example:

$$\begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix}^{-1} = \frac{1}{(1)(-2) - (3)(2)} \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix} = \frac{1}{-8} \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -1/4 & 3/8 \\ -1/8 & -1/4 \end{bmatrix}$$

(c) Properties include:

- $(A^{-1})^{-1} = A$
- $(AB)^{-1} = B^{-1}A^{-1}$
- $(\alpha A)^{-1} = \frac{1}{\alpha} A^{-1}$

5. Determinants:

(a) The **determinant** of a matrix, denoted $\det A$, is a number calculated from the matrix. We’ve seen this for $2 \times 2$ and $3 \times 3$ matrices.

(b) Properties include:

- A matrix $A$ has an inverse if and only if $\det A \neq 0$.
- For a $2 \times 2$ case $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$.
- We have $\det (A^{-1}) = 1/(\det A)$.

6. Minutia:

(a) The **transpose** of an $n \times n$ matrix $A$, denoted $A^T$, is found by reflecting the matrix in its main diagonal.

**Example:** We have:

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 4 & 7 \\ 8 & -1 & 5 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 8 \\ 2 & 4 & -1 \\ -3 & 7 & 5 \end{bmatrix}$$

(b) A matrix may have complex numbers in it, in which case its **complex conjugate** denoted either $\bar{A}$ or $A^c$, is found by taking the complex conjugate of each entry.

**Example:** We have:

$$\begin{bmatrix} 1 - 2i & 5 \\ 5 + i & 7 + 8i \end{bmatrix}^C = \begin{bmatrix} 1 + 2i & 5 \\ 5 - i & 7 - 8i \end{bmatrix}$$