1. Introduction:
   It’s far easier to manage systems of differential equations when we can rephrase them in the
   language of matrices and vectors. To that end, here are the essentials.

2. Basic Definitions:
   (a) A matrix is a rectangular array of numbers. It has size $m \times n$ if there are $m$ rows and $n$
columns. Matrices are typically denoted using capital letters:

   Example: Here is a $3 \times 4$ matrix:

   $$A = \begin{bmatrix}
   1 & 3 & -1 & 0 \\
   0 & 5 & 0 & 17 \\
   2 & 2 & -7 & 3 \\
   \end{bmatrix}$$

   (b) Most of the matrices in this class will be square, meaning they have the same number of rows
   and columns. Mostly we’ll deal with $2 \times 2$ and $3 \times 3$ matrices.

   (c) The identity matrix $I_n$ is the square $n \times n$ matrix with 1s on the main diagonal (upper-left
to lower right) and 0’s elsewhere. When the size is clear from context we just write $I$.

   Example:

   $$I_3 = \begin{bmatrix}
   1 & 0 & 0 \\
   0 & 1 & 0 \\
   0 & 0 & 1 \\
   \end{bmatrix}$$

   (d) The zero matrix is matrix of all zeros.

   (e) A vector is a matrix which is a single column. Vectors are usually denoted in lower-case with
   a bar over the letter.

   Example: $\vec{a} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$
3. Combining Matrices and Vectors:

(a) We can add matrices and vectors by adding matching entries provided they both have the same size.

Example: For example:
\[
\begin{bmatrix}
1 & 2 \\
3 & 4
\end{bmatrix} + \begin{bmatrix}
-1 & 0 \\
6 & 2
\end{bmatrix} = \begin{bmatrix}
1-1 & 2+0 \\
3+6 & 4+2
\end{bmatrix} = \begin{bmatrix}
0 & 2 \\
9 & 8
\end{bmatrix}
\]

(b) We can multiple an \( n \times n \) matrix \( A \) by a vector \( \bar{x} \) with \( n \) entries to get a new vector with \( n \) entries. We do this by multiplying each row of the matrix by the vector (element by element and add). This is easier to see:

Example: We have:
\[
\begin{bmatrix}
1 & 2 & -3 \\
0 & 4 & 7 \\
8 & -1 & 5
\end{bmatrix} \begin{bmatrix}
5 \\
3 \\
2
\end{bmatrix} = \begin{bmatrix}
(1)(5) + (2)(3) + (-3)(2) \\
(0)(5) + (4)(3) + (7)(2) \\
(8)(5) + (-1)(3) + (5)(2)
\end{bmatrix} = \begin{bmatrix}
5 \\
26 \\
47
\end{bmatrix}
\]

(c) We can multiple an \( n \times n \) matrix by another \( n \times n \) matrix by multiplying the first matrix by each of the columns in the second matrix as if it were just a vector, then taking these new vectors an putting them together in a new matrix.

Example: Here it is with lots of brackets to help you see what’s going on:
\[
\begin{bmatrix}
2 & 1 \\
4 & 3
\end{bmatrix} \begin{bmatrix}
-1 & 3 \\
5 & 9
\end{bmatrix} = \begin{bmatrix}
2 & 1 & -1 & 3 \\
4 & 3 & 5 & 9
\end{bmatrix} \begin{bmatrix}
2 & 1 \\
4 & 3
\end{bmatrix} \begin{bmatrix}
3 \\
9
\end{bmatrix} = \begin{bmatrix}
(2)(-1) + (1)(5) \\
(4)(-1) + (3)(5)
\end{bmatrix} \begin{bmatrix}
(2)(3) + (1)(9) \\
(4)(3) + (3)(9)
\end{bmatrix} = \begin{bmatrix}
3 & 15 \\
9 & 39
\end{bmatrix}
\]

(d) It’s almost always the case that for matrices \( A \) and \( B \) that \( AB \neq BA \).

(e) The identity matrix acts like the number 1 in that for any matrix \( A \) we have:
\[AI = IA = A\]
4. Inverses:

(a) The *inverse* of an \( n \times n \) matrix \( A \) is another matrix denoted \( A^{-1} \) such that \( AA^{-1} = A^{-1}A = I \). It’s like a “reciprocal” for matrices.

(b) For the \( 2 \times 2 \) size there is a formula:

\[
\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}
\]

**Example:** For example:

\[
\begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix}^{-1} = \frac{1}{(1)(-2) - (3)(2)} \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix} = \frac{1}{-8} \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -1/4 & 3/8 \\ -1/8 & -1/4 \end{bmatrix}
\]

(c) Properties include:

- \( (A^{-1})^{-1} = A \)
- \( (AB)^{-1} = B^{-1}A^{-1} \)
- \( (\alpha A)^{-1} = \frac{1}{\alpha} A^{-1} \)

5. Determinants:

(a) The *determinant* of a matrix, denoted \( \text{det} \ A \), is a number calculated from the matrix. We’ve seen this for \( 2 \times 2 \) and \( 3 \times 3 \) matrices.

(b) Properties include:

- A matrix \( A \) has an inverse if and only if \( \text{det} \ A \neq 0 \).
- For a \( 2 \times 2 \) case \( \text{det} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc \).
- We have \( \text{det} \ (A^{-1}) = 1/(\text{det} \ A) \).

6. Minutia:

(a) The *transpose* of an \( n \times n \) matrix \( A \), denoted \( A^T \), is found by reflecting the matrix in its main diagonal.

**Example:** We have:

\[
\begin{bmatrix} 1 & 2 & -3 \\ 0 & 4 & 7 \\ 8 & -1 & 5 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 8 \\ 2 & 4 & -1 \\ -3 & 7 & 5 \end{bmatrix}
\]

(b) A matrix may have complex numbers in it, in which case its *complex conjugate* denoted either \( A^\ast \) or \( A^c \), is found by taking the complex conjugate of each entry.

**Example:** We have:

\[
\begin{bmatrix} 1 - 2i & 5 \\ 5 + i & 7 + 8i \end{bmatrix}^C = \begin{bmatrix} 1 + 2i & 5 \\ 5 - i & 7 - 8i \end{bmatrix}
\]