MATH 246: Exam 3 Sample 2

1. Rewrite the following second-order differential equation as a system of first-order:

\[ 2D^2 y + tDy - 5y = \sin t \]

2. Find a fundamental pair of solutions to the system:

\[ \bar{x}' = \begin{bmatrix} 5 & -2 \\ 2 & 1 \end{bmatrix} \bar{x} \]

3. Solve the initial value problem

\[ \begin{align*}
    x_1' &= 2x_1 + 2x_2 & x_1(0) &= 2 \\
    x_2' &= 3x_1 + 7x_2 & x_2(0) &= -1
\end{align*} \]

4. Write down the differential equation corresponding to the following tank diagram. Assume the tanks start (do they remain?) full.

5. Consider the system:

\[ \bar{x}' = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix} \bar{x} \]

A fundamental matrix for this system is:

\[ \Psi(t) = \begin{bmatrix}
    1 & 2t + 1 \\
    1 & 2t
\end{bmatrix} \]

(a) Find the natural fundamental matrix \( \Phi(t) \) associated to \( t_I = 1 \).

(b) Use your answer to (a) to find the solutions to the IVP having \( \bar{x}(1) = \begin{bmatrix} 2 \\ -3 \end{bmatrix} \).

6. Use Hamiltonian Methods to sketch a family of solutions to the system:

\[ \begin{align*}
    x' &= 2x^2 + 2y \\
    y' &= -1 - 4xy
\end{align*} \]

7. Consider the competing species model given here:

\[ \begin{align*}
    x' &= (20 - 2x - y)x \\
    y' &= (20 - 2y)y
\end{align*} \]

This has stationary solutions \((0,0), (10,0), (0,10)\) and \((5,10)\). The first three of these have:

\[ \begin{align*}
    \partial^2 \bar{F}(0,0) &= \begin{bmatrix} 20 & 0 \\ 0 & 20 \end{bmatrix} \text{ with eigenpairs } \begin{bmatrix} 20 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
    \partial^2 \bar{F}(10,0) &= \begin{bmatrix} -20 & -10 \\ 0 & 20 \end{bmatrix} \text{ with eigenpairs } \begin{bmatrix} 20 \\ -4 \end{bmatrix} \text{ and } \begin{bmatrix} -20 \\ 1 \end{bmatrix} \\
    \partial^2 \bar{F}(0,10) &= \begin{bmatrix} 10 & 0 \\ 0 & -20 \end{bmatrix} \text{ with eigenpairs } \begin{bmatrix} -20 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 10 \\ 1 \end{bmatrix}.
\end{align*} \]

(a) Find \( \partial^2 \bar{F}(5,10) \) and its eigenvalues (not eigenvectors).

(b) Sketch a reasonable family of solutions.

(c) Explain in real-world terms what would happen to an initial population of \((4,0.1)\).

(d) Explain in real-world terms what would happen to an initial population of \((100,0.1)\).