MATH 246: Chapter 1 Section 1: Introduction to First-Order DEs Justin Wyss-Gallifent
Main Topics:

- Overview of first-order (O)DEs.
- Explicit first-order DEs.
- General solutions, initial value problems, specific (particular) solutions.
- Underlying theory regarding existence of solutions on an interval.

1. Introductory overview of first-order ODEs.
(a) A first-order ODE (not necessarily linear) is permitted to have an unknown function $y$ (of a single variable, say $t$ ) its derivative $y^{\prime}$ and then some other functions of $t$.
Example: $t\left(y^{\prime}\right)^{2}+y=\sin t$
Example: $y^{\prime}-t y=e^{2 t}$
Example: $\sin \left(y^{\prime}\right)+e^{y^{\prime}}=t$
(b) In general these can be very hard! The first step is always algebra though, basically we first solve for $y^{\prime}$ and then proceed from there. Thus for the next few sections we'll assume that we've solved for $y^{\prime}$ in terms of $t$ and $y$ and we'll focus on DEs that have the form $y^{\prime}=f(t, y)$.
That $f$ might be confusing, it's not the unknown function but rather it just represents the fact that we can have a bunch of $y$ and $t$ on the right hand side. In other words things like this:
Example: $y^{\prime}=t y$
Example: $y^{\prime}=4 t-8 y$
Example: $y^{\prime}=\frac{y}{t}$.
2. Explicit first-order DEs.
(a) Because solving even first-order ODEs is hard we'll go down even further and look at explicit first-order ODEs that have the form $y^{\prime}=f(t)$.
Example: $y^{\prime}=t^{2}$.
Example: $y^{\prime}=4 t+\sin t$.
(b) At this point you might have an epiphany and realize that often you can solve these because solving these is as easy as integrating the right side.
Example: $y^{\prime}=t^{2}$. To solve this we integrate to get $y=\frac{1}{3} t^{2}+C$ for any constant $C$.
Example: $y^{\prime}=4 t+\sin t$. To solve this we integrate to get $y=2 t^{2}-\cos t+C$ for any constant $C$.
3. General solutions, initial value problems, specific (particular) solutions
(a) We've started to notice that we can have many solutions to a DE. In the explicit DEs above get a constant $C$ which can be anything.
(b) A general solution to a DE is a solution involving constants and for which different constants will give all solutions.
(c) A specific solution or a particular solution is a solution in which a specific (particular) choice of constant(s) has been made.
Example: The general solution to $y^{\prime}=t^{2}$ is $y=\frac{1}{3} t^{3}+C$. Some specific solutions are $y=\frac{1}{3} t+1, y=\frac{1}{3} t-107$ and $y=\frac{1}{3} t+\pi$.
(d) Often when we encounter a DE it comes pre-packaged with an initial value, or IV. In our simple exact case (and in many future cases) this will be an insistance that $y\left(t_{I}\right)=y_{I}$ for specific $t_{I}$ and $y_{I}$. The DE and the IV together form an initial value problem or IVP. It's very common that $t_{I}=0$ but this isn't always the case!
Example: $y^{\prime}=2 t$ with $y(0)=3$ is an IVP.
Example: $y^{\prime}=2 t$ with $y(0)=5$ is an IVP with the same DE but different IV.
Example: $y^{\prime}=2 t$ with $y(1)=3$ is an IVP with again the same DE but different IV.
(e) When we're given an IVP the idea will be to first solve the DE to get the general solution and then use the IV to get the specific solution.

Example: $y^{\prime}=2 t$ with $y(0)=3$. First we find $y=t^{2}+C$, the general solution, and then $y(0)=0^{2}+C=3$ so $C=3$ and the specific solution is $y=t^{2}+3$.
Example: $y^{\prime}=2 t$ with $y(1)=3$. First we find $y=t^{2}+C$, the general solution, and then $y(1)=1^{2}+C=3$ so $C=2$ and the specific solution is $y=t^{2}+2$.
4. Intervals of Existence and Theory for Explicit IVPs:

We now know that solving the explicit DE given by $y^{\prime}=f(t)$ is as easy (or hard) as finding an antiderivative for $f(t)$. However the Fundamental Theorem of Calculus tells us something interesting. It states that if a function is continuous on an open interval then it is has an antiderivative on that open interval. This means that even if we can't actually find the antiderivative of $f(t)$ using techniques that we know, we still know that it exists, and therefore that there is a solution on an open interval as long as $f(t)$ is continuous on that open interval.

Example: Consider the explicit DE given by $y^{\prime}=t$. Since the function $t$ is continuous on $(-\infty, \infty)$ we know it has an antiderivative on $(-\infty, \infty)$ and therefore the DE has a solution there. In this case the general solution is $y=\frac{1}{2} t^{2}+C$.
Example: Consider the explicit DE given by $y^{\prime}=\frac{1}{t^{2}}$. The function $\frac{1}{t}$ is continuous on $(-\infty, 0)$ and on $(0, \infty)$. What this means is that it has solutions on each of those intervals.

When it comes to an explicit IVP we start with $y=f(t)$ and $y\left(t_{I}\right)=y_{I}$. We say that the interval of existence is the largest open interval containing $t_{I}$ on which a solution exists. This is found by find the largest open interval containing $t_{I}$ on which $f(t)$ is continuous.

Example: $y^{\prime}=\frac{1}{t^{2}}$ with $y(1)=5$. We notice the largest open interval containing $t_{I}=1$ on which $\frac{1}{t^{2}}$ is defined is $(0, \infty)$ and so this is the IE. Notice that we don't need to solve it, but we could, since the general solution is $y=-\frac{1}{t}+C$ and then $y(1)=-1+C=5$ so $C=4$ and the specific solution is $y=-\frac{1}{t}+4$.
Example: $y^{\prime}=\frac{t}{(t-3)(t+6)}$ with $t(0)=17$. We notice the largest open interval containing $t_{I}=0$ on which $\frac{t}{(t-3)(t+6)}$ is defined is $(-6,3)$ so this is the IE. We could possibly solve this with some messy partial fractions but we won't. However we do know for sure that there is a solution on this interval.

