Main Topics:

- Linear First-Order DEs and Linear Normal Form.
- General approach.
- Initial Value Problems.
- Theory

1. Linear first-order ODEs.

Recall that these will all have the form \( a_1(t)y' + a_0(t)y = c(t) \) where \( a_1, a_0, c \) can be any functions of \( t \).

**Example:** \( 4y' + 5y = 0 \)

**Example:** \( 4ty' + e^t y = \sin t \)

2. Linear Normal Form.

(a) Introduction:

We will usually divide through by \( a_1(t) \) and re-label a bit to get what is known as the linear normal form:

\[ y' + a(t)y = f(t) \]

for functions \( a(t) \) and \( f(t) \)

(b) General Solution.

These we can actually handle, and most of you did in Calculus II though it may be rusty.

Method: If we let \( A(t) \) be an antiderivative (any antiderivative, meaning use \(+0\) for the constant) of \( a(t) \) so that \( A'(t) = a(t) \) then observe:

\[
\begin{align*}
y' + a(t)y &= f(t) \\
e^{A(t)}y' + e^{A(t)}a(t)y &= f(t)e^{A(t)} \\
\frac{d}{dt}(e^{A(t)}y) &= f(t)e^{A(t)} \\
e^{A(t)}y &= \int f(t)e^{A(t)}
\end{align*}
\]

\[ y = e^{-A(t)}\int f(t)e^{A(t)}
\]

The only step that might concern you here is from line 2 to line 3. This is just the reverse of the product rule with a bit of chain rule thrown in. Reading it from line 3 to line 2 might be easier. We’ll see this sort of thing happen again with exact DEs later.

This process can either be repeated for each problem or treated simply as a recipe, meaning find \( A(t) \) and then use the formula at the end of the calculation.

Be careful though, the \( e^{-A(t)} \) is multiplied by the entire integral, meaning the \(+C\) too when you integrate.

We’ll call this final expression the integral-form solution:

\[ y = e^{-A(t)}\int f(t)e^{A(t)}
\]
Example: Consider $y' + 5y = 2$. We see that $a(t) = 5$ so $A(t) = 5t$ and the solution is
\[ y = e^{-5t} \int 2e^{5t} \, dt \]
\[ = e^{-5t} \left( \frac{2}{5} e^{5t} + C \right) \leftarrow \text{Note the parentheses!!!} \]
\[ = \frac{2}{5} + Ce^{-5t} \]
If you’d have forgotten the parentheses you’d have got an incorrect answer which I won’t even write here!

Example: Consider $ty' + 2y = t^4$ with $t > 0$. This is not in linear normal form so we divide by $t$ to get $y' + \frac{2}{t}y = t^3$. Then $a(t) = \frac{2}{t}$ so $A(t) = 2 \ln t$ and the solution is
\[ y = e^{-2 \ln t} \int t^3 e^{2 \ln t} \, dt \]
\[ = t^{-2} \int t^5 \, dt \]
\[ = t^{-2} \left( \frac{1}{6} t^6 + C \right) \]
\[ = \frac{1}{6} t^4 + C \frac{1}{t^2} \]

Here’s one with an IVP:

Example: Consider $y' - 6y = e^t$ with $y(0) = 2$. We see that $a(t) = -6$ so $A(t) = -6t$ and the general solution is
\[ y = e^{-(6t)} \int e^t e^{-6t} \, dt \]
\[ = e^{6t} \int e^{-5t} \, dt \]
\[ = e^{6t} \left( -\frac{1}{5} e^{-5t} + C \right) \]
\[ = -\frac{1}{5} e^t + Ce^{6t} \]
At this point $y(0) = -\frac{1}{5} e^0 + Ce^0 = -\frac{1}{5} + C = 2$ so that $C = \frac{11}{5}$ so the specific solution is
\[ y = -\frac{1}{5} e^t + \frac{11}{5} e^{6t} \]
At this point you can probably see that solving a first-order linear ODE is as easy (or as hard) as first finding $A(t)$ and then finding $\int f(t)e^{A(t)} \, dt$. 
(c) Note about the choice of $A(t)$. You might wonder what happened if you didn’t choose $+0$ as your constant when choosing $A(t)$. In fact it makes no difference. Suppose we took $A(t)$ and adjusted it by adding some number like $+7$. The solution would then be:

$$y = e^{-(A(t)+7)} \int f(t)e^{A(t)+7} \, dt = e^{-7}e^{-A(t)} \int f(t)e^{A(t)} \, dt = e^{-A(t)} \int f(t)e^{A(t)} \, dt$$

which is exactly the same.

3. Theory!

The Second Fundamental Theorem of Calculus states that if a function is continuous on an open interval then it has an antiderivative on that interval and that antiderivative will be continuous. What this means is that if $a(t)$ is continuous then $A(t)$ will exist and therefore so will $e^{-A(t)}$ and then provided that $f(t)$ is continuous then so will $\int f(t)e^{A(t)} \, dt$.

Warning! This doesn’t mean that these things are easy to calculate, just that they exist!

What this means is that if we have an initial value $y(t_I) = y_I$ then the interval of existence of the solution will be the largest open interval containing $t_I$ on which both $f(t)$ and $a(t)$ are continuous. As before this lets us find the IE even when we can’t solve the IVP.

**Example:** Consider $y' + \frac{1}{t^2}y = \frac{1}{t-5}$ with $y(2) = 17$. Here $a(t) = \frac{1}{t^2}$ and $f(t) = \frac{1}{t-5}$.

The largest open interval containing $t_I = 2$ on which both are continuous is $(0, 5)$ so this is the IE of the solution. Finding the solution is a different matter entirely but it exists on $(0, 5)$!

4. Integration Comment.

As a final note observe that there are two antiderivatives involved in the problem, finding $A(t)$ and finding $\int f(t)e^{A(t)} \, dt$. This latter one will often involve simplification involving $e$ and $\ln$ as well as substitution and integration by parts.