MATH 246: Chapter 1 Section 3: Separable DEs Justin Wyss-Gallifent
Main Topics:

- Separable DE and Method of Solution.
- Implicit vs. Explicit Solutions.
- Constant Solutions.
- Autonomous DEs.
- Effect of Initial Values on Solution Choice.
- Non-uniqueness of Solutions.
- Method Notation Note.

1. Separable ODEs.

A DE is separable if it can be written in the form $y^{\prime}=f(t) g(y)$. The word separable comes from the fact that the right side is separated into a product of a function of $t$ and a function of $y$.
Example: $y^{\prime}=t y$ is separable - it is already separated!
Example: $t y^{\prime}+y^{\prime}=y^{2}$ is separable because it can be separated, first by factoring $y^{\prime}(t+1)=y^{2}$ and then dividing $y^{\prime}=\frac{y^{2}}{t+1}$ and thinking of it as $y^{\prime}=\left(\frac{1}{t+1}\right) y^{2}$.
Example: $y^{\prime}=t+y$ is not separable. There is no way to write the right side as a product of a function of $t$ and a function of $y$.
2. Method: Solving for Non-Constant Solutions.

The method of solution for non-constant solutions is really slick:

$$
\begin{aligned}
y^{\prime} & =f(t) g(y) \\
\frac{d y}{d t} & =f(t) g(y) \\
\frac{1}{g(y)} d y & =f(t) d t \\
\int \frac{1}{g(y)} d y & =\int f(t) d t
\end{aligned}
$$

Where the integral on the left is with respect to $y$ and the integral on the right is with respect to $t$. Since both indefinite integrals should get their own constant, instead we just put a single $+C$ on the right.
The $\frac{1}{g(y)}$ looks really icky to integrate but in our examples it generally works out pretty nicely because it's often not a quotient at all.
Side note: We're really taking a stick and beating the notation into a pulp here. If this weird approach bothers you I've attached a note at the end explaining why it's simply a shorthand notation for something more rigorous.

Example: Consider $y^{\prime}=\frac{t}{y^{2}}$.
Note that you could think of this as $y^{\prime}=t\left(\frac{1}{y^{2}}\right)$. The key is to get all the $y$ s together multiplied by the $y^{\prime}$ on the left and leave all the $t$ s together on the right. We work as follows, multiplying by $y^{2}$ first, or if you prefer to think of it that way, dividing by $1 / y^{2}$ :

$$
\begin{aligned}
y^{\prime} & =\frac{t}{y^{2}} \\
y^{2} \frac{d y}{d t} & =t \\
y^{2} d y & =t d t \\
\int y^{2} d y & =\int t d t \\
\frac{1}{3} y^{3} & =\frac{1}{2} t^{2}+C \\
y & =\sqrt[3]{\frac{3}{2} t^{2}+3 C}
\end{aligned}
$$

There's an argument to be made at this point that since $3 C$ is just a constant we could write $C$ instead. However this results in a problem having two different $C$ s meaning two different things and given that we will often solve for this $C$ it's far safer to just leave it as is.

## 3. Observation: Constant Solutions:

When we solve a separable DE we do often actually divide by some $g(y)$ and in that case we have to independently look at the possiblity that $g(y)=0$. This may lead to constant functions which are additional solutions to the separable DE.

Example: Consider $y^{\prime}=e^{t} \sqrt{1-y^{2}}$.
The method of solution above goes as follows:

$$
\begin{aligned}
\frac{d y}{d x} & =e^{t} \sqrt{1-y^{2}} \\
\frac{1}{\sqrt{1-y^{2}}} d y & =e^{t} d t \\
\int \frac{1}{\sqrt{1-y^{2}}} d y & =\int e^{t} d t \\
\sin ^{-1} y & =e^{t}+C \\
y & =\sin \left(e^{t}+C\right)
\end{aligned}
$$

There is nothing wrong with this provided $\sqrt{1-y^{2}} \neq 0$. So what if $\sqrt{1-y^{2}}=0$ ? This would arise if $y= \pm 1$ and in fact these are completely valid solutions (functions!) to the DE . Thus overall the DE has two constant solutions as well as the nonconstant solutions!

All Solution: $y=-1, y=1, y=\sin \left(e^{t}+C\right)$
Conclusion: When we divide by some $g(y)$ the functions arising from when $g(y)=0$ are valid constant solutions to the DE and must be included in our final list.
4. Implicit versus Explicit Solutions:

It's entirely possible that when we solve a separable DE we are unable to solve for $y$ at the end, or it may be very difficult.

Example: Consider $y^{\prime}=\frac{t}{e^{y}+1}$.
Notice there are no constant solutions here because to solve it we do not divide by something which can be zero.
We work as follows:

$$
\begin{aligned}
\frac{d y}{d t} & =\frac{t}{e^{y}+1} \\
\left(e^{y}+1\right) d y & =t d t \\
\int e^{y}+1 d y & =\int t d t \\
e^{y}+y & =\frac{1}{2} t^{2}+C
\end{aligned}
$$

In this case it's reasonable to stop here and say that we have implicitly defined the solutions An implicit solution is a solution in which we have not actually achieved $y=$.
Ideally of course we would be able to solve for $y$, this would yield an explicit solution.
5. Autonomous ODEs:

There is a special kind of separable ODE called autonomous. This occurs when $f(t)=1$ and so instead we have $y^{\prime}=g(y)$. This can be solved like any other separable ODE. We only mention it because these will arise repeatedly over the course in various places.
Example: Consider $y^{\prime}=(y-4)^{2}$.
Here $g(y)=(y-4)^{2}$ which equals 0 when $y=4$ so this is the constant solution. The nonconstant solutions we obtain as follows:

$$
\begin{aligned}
\frac{d y}{d t} & =(y-4)^{2} \\
(y-4)^{-2} d y & =1 d t \\
\int(y-4)^{-2} d y & =\int 1 d t \\
-(y-4)^{-1} & =t+C \\
(y-4)^{-1} & =-(t+C) \\
(y-4) & =\frac{-1}{t+C} \\
y & =\frac{-1}{t+C}+4
\end{aligned}
$$

6. Two Small Initial Value Notes:
(a) Choosing Solutions:

When we solve a separable ODE and get an implicit solution for which there seems to be more than one explicit solution, an initial value usually tells us which one it is:
Example: Consider $y^{\prime}=\frac{t}{y}$ with $y(1)=-3$.
First we solve the DE:

$$
\begin{aligned}
\frac{d y}{d t} & =\frac{t}{y} \\
y d y & =t d t \\
\int y d y & =\int t d t \\
\frac{1}{2} y^{2} & =\frac{1}{2} t^{2}+C \\
y^{2} & =t^{2}+2 C \\
y & = \pm \sqrt{t^{2}+2 C}
\end{aligned}
$$

We see that there are two explicit solutions to the DE.
When we consider the initial value we have $y(1)= \pm \sqrt{1^{2}+2 C}=-3$ so we are forced to use the negative in front of the square root. Thus $-\sqrt{1^{2}+2 C}=-3$ so $1+2 C=9$ and $C=4$. Then the explicit solution is $y=-\sqrt{t^{2}+8}$.
Note: There are no constant solutions here since $g(y)=\frac{1}{y}$ is never 0 .
(b) Uniqueness (?) of Solutions:

The existence of constant solutions often leads to non-unique solutions to IVPs. This can happen when a constant solution satisfies the DE but also the procedural method gives another solution. The way to manage this is to not forget to find your constant solutions and check if they satisfy the IV.
Example: Consider $y^{\prime}=y^{2 / 3}$ with $y(0)=0$. Notice that $y=0$ is a constant solution which also satisfies the DE. However the DE is separable:

$$
\begin{aligned}
\frac{d y}{d t} & =y^{2 / 3} \\
y^{-2 / 3} d y & =1 d t \\
\int y^{-2 / 3} d y & =\int 1 d t \\
3 y^{1 / 3} & =t+C \\
y & =\left(\frac{1}{3} t+\frac{1}{3} C\right)^{3}
\end{aligned}
$$

Then $y(0)=\left(\frac{1}{3} C\right)^{3}=0$ so $C=0$. This gives the additional solution $y=\left(\frac{1}{3} t\right)^{3}=\frac{1}{27} t^{3}$.

## 7. Overlap

At this juncture it might be helpful to notice that an ODE doesn't need to be just one of the categories we've looked at - explicit, first-order linear, and separable - it could fall into more than one category.
Example: $y^{\prime}=t y$ is both separable and first-order linear.
Example: $y^{\prime}=t^{2}$ is all of explicit, separable and first-order linear.
8. Justification for the strange approach.

It may bother you that we can rip apart the DE like we do and simply stick an integral sign on both sides. Really this is just shorthand notation for a more rigorous approach. More rigorously once our DE is rewritten as:

$$
G(y(t)) y^{\prime}(t)=f(t)
$$

Since they are equal we can integrate bot sides with respect to $t$ :

$$
\int G(y(t)) y^{\prime}(t) d t=\int f(t) d t
$$

On the left we make the substitution $u=y(t)$ which yields $\frac{d u}{d t}=y^{\prime}(t)$ which we then substitute in:

$$
\int G(u) \frac{d u}{d t} d t=\int f(t) d t
$$

This then simplifies to:

$$
\int G(u) d u=\int f(t) d t
$$

