MATH 246: Chapter 1 Section 5: Graphical Methods Justin Wyss-Gallifent
Main Topics:

- Quantitative versus Qualitative Solutions.
- Phase-Line Portraits and Sketching Autonomous Systems.
- Contour Plots for Implicit Solutions.
- Slope Fields.

1. Introduction:

The overarching goal of this section is to find things out about solutions to DEs without actually solving them explicitly. Instead we attack them graphically.
2. Phase Line Portraits for Autonomous Differential Equations.
(a) Introductory Example.

Consider the autonomous $\mathrm{DE} y^{\prime}=y(y-3)$. This has constant solutions $y=0$ and $y=3$. But what about other solutions? Well the DE tells us that if we know $y$ then we know $y^{\prime}$. A sign chart giving information about $y^{\prime}$ can give us a wealth of information:


Consider now that:

- A solution below $y=0$ will be increasing. Moreover for larger positive $t$-values it will get closer to $y=0$ where it will level off and for larger negative $t$-values it will get closer to $\infty$.
- A solution between $y=0$ and $y=3$ will be decreasing. Moreover for larger positive $t$-values it will get closer to $y=0$ where it will level off and for larger negative $t$-values it will get closer to $y=3$ where it will level off.
- A solution above $y=3$ will be increasing. Morevoer for larger positive $t$-values it will get closer to $\infty$ and for larger negative $t$-values it will get closer to $y=3$ where it will level off.
Basically all possible solutions will look like these:

(b) General Understanding:

An autonomous DE will generally have constant solutions. Between the constant solutions is where interesting things happen. To see what's going on between the constant solutions draw a number line which gives information about whether solutions are increasing or decreasing.
This number line is the phase line portrait of the autonomous differential equation.
Solutions approaching constant solutions will do so asymptotically. This is true both for larger positive $t$-values and larger negative $t$-values.
Moreover this gives some information about those constant solutions:
i. If nearby solutions move away from the constant solution on both sides then the constant solution is unstable.
ii. If nearby solutions move toward the constant solution on both sides then the constant solution is stable.
iii. If there is different behavior on each side then the constant solution is semistable.

Example: Consider $y^{\prime}=y(y-3)(y+2)^{2}$. This has the following sign chart:


Basically all possible solutions will look like these:


From these families of solutions we can draw all sorts of conclusions:

- The constant solution $y=-2$ is stable, the constant solution $y=0$ is unstable, and the constant solution $y=3$ is semistable.
- The particular solution $y(t)$ satisfying $y(0)=\alpha$ has $\lim _{t \rightarrow \infty} y(t)=2$ when $-\infty<\alpha<0$.
- The particular solution $y(t)$ satisfying $y(0)=\alpha$ is decreasing when $-2<\alpha<0$.
- $y=-2$ is stable.
- $y=0$ is unstable.
- $y=3$ is semistable.


## 3. Contour Plots of Implicit Solutions

(a) When we solve a separable DE we often get an implicit solution with a $C$ in it. This implicit solution is an equation. If we pick various values of $C$ and plot the resulting equations we get a contour plot.
What's useful about these contour plots is that the parts of the curves that form functions are explicit solutions to the DE because they're functions ( $y$ in terms of $t$ ) that satisfy the implicit solution. This means that we can pick a point on a curve and follow it as far left and right as possible and the result is the graph of an explicit solution to the DE.
Example: Consider $\frac{d y}{d x}=\frac{1}{2 y-6}$. This is separable with general solution $y^{2}-6 y=x+C$. This is not as bad as it looks:

$$
\begin{aligned}
y^{2}-6 y & =x+C \\
y^{2}-6 y+9 & =x+C+9 \\
(y-3)^{2} & =x+C+9
\end{aligned}
$$

These are all parabolas opening right with their vertices at $y=3$. If we sketch a few of these (note that they extend out forever, this is just a subset):


Solutions to the DE are functions (must pass the vertical line test!) which lie along these parabolic curves. For example the specific solution satisfying $y(0)=0$ corresponds to $C=0$ and looks like this:


From this contour plot we can draw all sorts of conclusions:

- Solutions extend infinitely far to the right but not the left.
- Solutions are either increasing or decreasing but not both.
- The specific solution $y(t)$ with $y(0)=0$ is a decreasing function with $\lim _{t \rightarrow \infty} y(t)=-\infty$.
- The specific solution with $y(0)=5$ is an increasing function.
- Solutions are either always increasing or always decreasing.

4. Direction (Slope) Fields
(a) As a last-ditch effort any $\frac{d y}{d t}=f(t, y)$ (any first-order) is essentially telling us the slope of a solution at a point. Consequently we can plug in lots of $t$ and $y$ and indicate what the slope would be of a solution passing through that point. The result is a direction field or slope field. Then we can trace functions which follow the field and draw conclusions.
(b) Example: Here is the direction field for $\frac{d y}{d t}=t-y^{2}$, a hard DE to solve:


Solutions to the DE are functions (must pass the vertical line test!) which follow these arrows. For example the specific solution satisfying $y(-1)=0$ looks like this:


From this direction field we can draw all sorts of conclusions:

- The specific solution satisfying $y(-1)=0$ has a relative minimum approximately ( $0,-0.5$ ).
- We can observe categories, for example not all solutions have relative mimima.

Note however that we're somewhat restricted by the range we drew!

