MATH 246: Chapter 1 Section 8: Exact Differential Equations
Justin Wyss-Gallifent

Main Topics:

- Exact Differential Equations.
- Method of Solution.
- Integrating Factors.

---

1. **A Bit of History and Introduction:** Suppose $H(x,y)$ is a function and y is a function of x. Then by the chain rule we know $\frac{d}{dx}H(x,y) = H_x(x,y) + H_y(x,y)\frac{dy}{dx}$.

   So now consider the following differential equation:

$$3x^2y^2 + 2x^3y\frac{dy}{dx} = 0$$

   You may notice that the left side looks like the result of the chain rule and is actually so, when $H(x,y) = x^3y^2$. Don't worry about if there's a formal method for where $H(x,y)$ comes from for now, just notice that $H_x(x,y) = 3x^2y^2$ and $H_y(x,y) = 2x^3y$. What this means is that the differential equation may be rewritten by undoing the chain rule on the left:

$$\underbrace{3x^2y^2 + 2x^3y\frac{dy}{dx}}_{\frac{d}{dx}[x^3y^2]} = 0$$

   So then when the derivative of something is zero, that thing is a constant:

$$\begin{aligned}\frac{d}{dx}\left[x^3y^2\right] &= 0\\ x^3y^2 &= C\end{aligned}$$

   and we've solved it, at least implicitly!

2. **Definition and Method:** A differential equation is *exact* if it has the form:

$$H_x(x,y) + H_y(x,y)\frac{dy}{dx} = 0$$

   for some function $H(x,y)$. When a differential equation is exact, solving implicitly is as easy as finding $H(x,y)$ and setting $H(x,y) = C$ for any constant.

   Here are a few exact differential equations. For each, $H(x,y)$ is written in the middle and the implicit solution to the right.

Exact DE	$H(x,y)$	Solution to DE
$y + x\frac{dy}{dx} = 0$	$H(x,y) = xy$	$xy = C$
$y + (x+2y)\frac{dy}{dx} = 0$	$H(x,y) = xy + y^2$	$xy + y^2 = C$
$\frac{1}{y} - \frac{x}{y^2}\frac{dy}{dx} = 0$	$H(x,y) = \frac{x}{y}$	$\frac{x}{y} = C$
$y\cos(xy) + x\cos(xy)\frac{dy}{dx} = 0$	$H(x,y) = \sin(xy)$	$\sin(xy) = C$

3. **Detecting Exactness and Finding H:** There is a trick to detecting whether a differential equation is exact. If the differential equation has the form:

$$M + N\frac{dy}{dx} = 0$$

then it is exact if and only if $M_y = N_x$. You can test all the ones above. Then you can check that this next one is not exact:

$$xy + y\frac{dy}{dx} = 0$$

In this case $M_y = x$ and $N_x = 0$. Not equal, not exact.

Once you know that your differential equation is exact, often you can guess at $H(x, y)$. However if you're struggling, there's a systematic method for finding it. Here's an example from above:

$$y + (x + 2y)\frac{dy}{dx} = 0$$

We want $H(x, y)$ with (A) $H_x(x, y) = y$ and (B) $H_y(x, y) = x + 2y$. Observe:

We want (A):	$H_x(x, y) = y$
This tells us that:	$H(x, y) = xy + h(y)$
From this line:	$H_y(x, y) = x + h'(y)$
But from (B):	$H_y(x, y) = x + 2y$
Set these equal:	$x + h'(y) = x + 2y$
Solve for $h'(y)$:	$h'(y) = 2y$
Find $h(y)$:	$h(y) = y^2 + D$
Put back into second line:	$H(x, y) = xy + y^2 + D$

We can choose any D so choose $D = 0$ to get $H(x, y) = xy + y^2$.

Example: Find $H(x, y)$ to solve $x + 1 + \frac{1}{y} - \frac{x}{y^2}\frac{dy}{dx} = 0$. Follow the exact procedure above, here we want (A) $H_x(x, y) = x + 1 + \frac{1}{y}$ and (B) $H_y(x, y) = -\frac{x}{y^2}$:

We want (A):	$H_x(x, y) = x + 1 + \frac{1}{y}$
This tells us that:	$H(x, y) = \frac{1}{2}x^2 + x + \frac{x}{y} + h(y)$
From this line:	$H_y(x, y) = -\frac{x}{y^2} + h'(y)$
But from (B):	$H_y(x, y) = -\frac{x}{y^2}$
Set these equal:	$-\frac{x}{y^2} + h'(y) = -\frac{x}{y^2}$
Solve for $h'(y)$:	$h'(y) = 0$
Find $h(y)$:	$h(y) = D$
Put back into second line:	$H(x, y) = \frac{1}{2}x^2 + x + \frac{x}{y} + D$

Then choose $D = 0$ to get $H(x, y) = \frac{1}{2}x^2 + x + \frac{x}{y}$ and the solution to our DE is $\frac{1}{2}x^2 + x + \frac{x}{y} = C$.

4. **Integrating Factors:** It's not uncommon to have a differential equation which is not quite exact but can be made exact by multiplying through by some function called an *integrating factor*. For example the differential equation

$$2y + x\frac{dy}{dx} = 0$$

is not exact because $M_y = 2$ and $N_x = 1$ so $M_y \neq N_x$. But if we multiply through by x we get the new differential equation

$$2xy + x^2\frac{dy}{dx} = 0$$

which is exact because $M_y = 2x$ and $N_x = 2x$. Now $H(x, y) = x^2y$ and the solution is $x^2y = C$.

The question is how to come up with this integating factor. This is very challenging so we'll look at two specific cases, either the integrating factor is a function $f(x)$ of only x or the integrating factor is a function $g(y)$ of only y.

5. **Examples:**

Example 1: Consider the diffential equation we've seen before:

$$2y + x\frac{dy}{dx} = 0$$

Here $M = 2y$ and $N = x$, these are different so it's not exact. Let's look for some $f(x)$ so that when we multiply through the result is exact:

$$2yf(x) + xf(x)\frac{dy}{dx} = 0$$

For this to be exact we'd need:

$$\begin{aligned}
[xf(x)]_x &= [2yf(x)]_y \\
1f(x) + xf'(x) &= 2f(x) + 2y(0) \\
xf'(x) &= f(x) \\
f'(x) &= \frac{f(x)}{x}
\end{aligned}$$

We can see that $f(x) = x$ does the job. This is then our integrating factor and we multiply our original differential equation through by it to get the exact differential equation

$$2xy + x^2\frac{dy}{dx} = 0$$

which has $H(x, y) = x^2y$ and hence solution $x^2y = C$.

Addendum: If we tried $g(y)$ we'd want this to be exact:

$$2yg(y) + xg(y)\frac{dy}{dx} = 0$$

This would mean:

$$\begin{aligned}
[xg(y)]_x &= [2yg(y)]_y \\
1g(y) + x(0) &= 2g(y) + 2yg'(y) \\
g'(y) &= -\frac{g(y)}{2y}
\end{aligned}$$

It's much harder to see what might work here. Interestingly $g(y) = y^{-1/2}$ will work, yielding the exact:

$$2y^{1/2} + xy^{-1/2}\frac{dy}{dx}$$

which has $H(x, y) = 2xy^{1/2}$ and hence solution $2xy^{1/2} = C$.

Example 2: Consider the differential equation

$$y + (x + xy)\frac{dy}{dx} = 0$$

Here $M = y$ and $N = x + xy$, these are different so it's not exact. Let's look for some $g(y)$ so that when we multiply through the result is exact:

$$yg(y) + (x + xy)g(y)\frac{dy}{dx} = 0$$

For this to be exact we'd need:

$$\begin{aligned}
[(x + xy)g(y)]_x &= [yg(y)]_y \\
(1 + y)g(y) &= 1g(y) + yg'(y) \\
g(y) + yg(y) &= g(y) + yg'(y) \\
g'(y) &= g(y)
\end{aligned}$$

We can see that $g(y) = e^y$ does the job. This is then our integrating factor and we multiply our original differential equation through by it to get the exact differential equation

$$ye^y + (x + xy)e^y\frac{dy}{dx} = 0$$

which has $H(x, y) = xye^y$ and hence solution $xye^y = C$.

Addendum: If we tried $f(x)$ we'd want this to be exact:

$$yf(x) + (x + xy)f(x)\frac{dy}{dx} = 0$$

This would mean:

$$\begin{aligned}
[(x + xy)f(x)]_x &= [yf(x)]_y \\
(1 + y)f(x) + (x + xy)f'(x) &= 1f(x) + y(0) \\
f(x) + yf(x) + (x + xy)f'(x) &= f(x) \\
yf(x) + (x + xy)f'(x) &= 0 \\
f'(x) &= -\frac{yf(x)}{(x + xy)}
\end{aligned}$$

It's not at all obvious if anything this works.