MATH 246: Chapter 2 Section 2: Homogeneous Equations - Method and Theory Justin Wyss-Gallifent

Main Topics:

- Definition of Homogeneous.
- Motivational Example for Second Order.
- Theory for Second and Third Order.
- 1. Introduction: Since even linear higher-order DEs are difficult we are going to simplify even more. For today we're going to look at *homogeneous* higher-order linear DEs, in which the forcing function f(t) is equal to 0. That is:

 $\begin{array}{ll} \mbox{First-Order} & y'+a(t)y=0\\ \mbox{Second-Order} & y''+a(t)y'+b(t)y=0\\ \mbox{Third-Order} & y'''+a(t)y''+b(t)y'+c(t)y=0\\ \mbox{:} & \mbox{:} \end{array}$

2. A Motivational Example: Consider the second-order homogeneous linear DE:

$$y'' - y' - 2y = 0$$

Next look at the following two functions, don't worry about where they came from:

$$Y_1(t) = e^{2t}$$
 and $Y_2(t) = e^{-t}$

We can easily see that these are both solutions to the DE by plugging them (and their derivatives) in and checking.

(a) Observation 1 - Getting More Solutions:

Notice that if we take a *linear combination* of these two, meaning

$$Y(t) = C_1 e^{2t} + C_2 e^{-t}$$

where C_1 and C_2 are constants. Then we can easily see that this is also a solution to the DE by plugging it (and its derivatives) in and checking.

(b) Observation 2 - Getting All Solutions:

We can build new solutions from these two but can we build all solutions this way? Well suppose that we had some solution to the DE, call it Y(t). What we want to know is if we can find C_1 and C_2 so that $Y(t) = C_1 e^{2t} + C_2 e^{-t}$ for this Y(t)?

Well, suppose we find that $Y(0) = y_0$ and $Y'(0) = y_1$. Since $Y'(t) = 2C_1e^{2t} - C_2e^{-t}$ we would need

$$y_0 = Y(0) = C_1 + C_2$$

 $y_1 = Y'(0) = 2C_2 - C_2$

Can we find such values? Since $\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} \neq 0$ there is a unique solution.

Notice now that since this is a solution to the IVP and since there is only one solution to the IVP this must be the solution we were looking for.

(c) Observation 3 - Anything Special About Those Two?

We can't just start with any two solutions. To see this observe that if we'd started with $Y_1(t) = e^{2t}$ and $Y_2(t) = 17e^{2t}$ that both of these are solutions. Again any linear combination $Y(t) = C_1e^{2t} + C_217e^{2t}$ is a solution. However is every solution to the DE a linear combination? Again, suppose Y(t) is a solution and $Y(0) = y_0$ and $Y'(0) = y_1$. Then $Y'(t) = 2C_1e^{2t} + 34C_2e^{2t}$ and we would need

$$y_0 = Y(0) = C_1 + 17C_2$$

 $y_1 = Y'(0) = 2C_1 + 34C_2$

Since $\begin{vmatrix} 1 & 17 \\ 2 & 34 \end{vmatrix} = 0$ there may be no solution. That is, we can't guarantee a solution.

3. Theory:

- (a) Theory for Second-Order Homogeneous: y'' + a(t)y' + b(t)y = 0
 - For a second-order homogeneous linear DE we need to find two solutions $Y_1(t)$ and $Y_2(t)$ with a special relationship. That relationship is that their *Wronskian* does not equal the zero function, where:

$$W[Y_1, Y_2] = \left| \begin{array}{cc} Y_1 & Y_2 \\ Y_1' & Y_2' \end{array} \right|$$

Alternately the two solutions cannot be multiples of each other. They form a fundamental set or fundamental pair of solutions $\{Y_1(t), Y_2(t)\}$.

- Every solution is then a linear combination of the fundamental pair. This means the general solution is $Y(t) = C_1 Y_1(t) + C_2 Y_2(t)$.
- A second-order IVP must provide $y(t_I)$ and $y'(t_I)$ in order to find the specific solution.
- This solution is unique on the *interval of existence and uniqueness* which is the largest open interval containing t_I on which a(t) and b(t) are differentiable.
- (b) Theory for Third-Order Homogeneous: y''' + a(t)y'' + b(t)y' + c(t)y = 0
 - For a third-order homogeneous linear DE we need to find three solutions $Y_1(t)$, $Y_2(t)$, and $Y_3(t)$ with a special relationship. That relationship is that their *Wronskian* does not equal the zero function, where:

$$W[Y_1, Y_2, Y_3] = \begin{vmatrix} Y_1 & Y_2 & Y_3 \\ Y_1' & Y_2' & Y_3' \\ Y_1'' & Y_2'' & Y_3'' \end{vmatrix}$$

Alternately it must be impossible to write one of the solutions as a linear combination of the others. They form a *fundamental set* of solutions $\{Y_1(t), Y_2(t), Y_3(t)\}$.

- Every solution is then a linear combination of the fundamental set. This means the general solution is $Y(t) = C_1Y_1(t) + C_2Y_2(t) + C_3Y_3(t)$.
- A third-order IVP must provide $y(t_I)$, $y'(t_I)$, and $y''(t_I)$ in order to find the specific solution.
- This solution is unique on the *interval of existence and uniqueness* which is the largest open interval containing t_I on which a(t) and b(t) and c(t) are differentiable.

(c) Theory for Higher-Order:

You can probably see the pattern.

(d) **Critical Note:** Don't worry about where these fundamental sets are coming from right now, just realize that we (somehow) need obtain them!

4. Practice for Both:

Here are some examples:

Example: Consider y'' + 4y = 0. First we'll show that $Y_1(t) = \sin(2t)$ and $Y_2(t) = \cos(2t)$ form a fundamental pair. We check they are solutions (omitted) and we check:

$$W[Y_1, Y_2] = \begin{vmatrix} \sin(2t) & \cos(2t) \\ 2\cos(2t) & -2\sin(2t) \end{vmatrix} = -2\sin^2(2t) - 2\cos^2(2t) = -2 \neq 0$$

This tells us that $Y_1(t)$ and $Y_2(t)$ form a fundamental pair and that the general solution is:

$$Y(t) = C_1 \sin(2t) + C_2 \cos(2t)$$

So now if we have the IVP with Y(0) = 4 and Y'(0) = 2 we can find the specific solution by first finding:

$$Y'(t) = 2C_1 \cos(2t) - 2C_2 \sin(2t)$$

and then solving the system:

$$4 = Y(0) = C_1 \sin(2(0)) + C_2 \cos(2(0)) = C_2$$

$$2 = Y'(0) = 2C_1 \cos(2(0)) - 2C_2 \sin(2(0)) = 2C_1$$

So that $C_1 = 1$ and $C_2 = 4$ and the specific solution is:

$$Y(t) = \sin(2t) + 4\cos(2t)$$

Example: Consider $(1 + t^2)y'' - 2ty' + 2y = 0$. First we'll show that $Y_1(t) = t$ and $Y_2(t) = t^2 - 1$ form a fundamental pair. Notice that it doesn't matter whether we divide by $1 + t^2$ or not when we check these. We check they are solutions (omitted) and we check:

$$W[Y_1, Y_2] = \begin{vmatrix} t & t^2 - 1 \\ 1 & 2t \end{vmatrix} = 2t^2 - (t^2 - 1) = t^2 + 1 \neq 0$$

This tells us that $Y_1(t)$ and $Y_2(t)$ form a fundamental pair and that the general solution is:

$$Y(t) = C_1 t + C_2 (t^2 - 1)$$

So now if we have the IVP with Y(2) = -5 and Y'(2) = 7 we can find the specific solution by first finding:

$$Y'(t) = C_1 + 2C_2 t$$

and then solving the system:

$$-5 = Y(2) = C_1(2) + C_2(2^2 - 1) = 2C_1 + 3C_2$$

7 = Y'(2) = C_1 + 2C_2(2) = C_1 + 4C_2

So that $C_1 = -\frac{41}{5}$ and $C_2 = -\frac{19}{5}$ and the specific solution is:

$$Y(t) = -\frac{41}{5}t + \frac{19}{5}(t^2 - 1)$$

Example: Consider $D^3y - 2D^2y = 0$ First we'll show that $Y_1(t) = 1$, $Y_2(t) = t$ and $Y_3(t) = e^{2t}$ form a fundamental set. We check they are solutions (omitted) and we check:

$$W[Y_1, Y_2, Y_3] = \begin{vmatrix} 1 & t & e^{2t} \\ 0 & 1 & 2e^{2t} \\ 0 & 0 & 4e^{2t} \end{vmatrix} = 4e^{2t} \neq 0$$

This tells us that $Y_1(t)$, $Y_2(t)$ and $Y_3(t)$ form a fundamental set and that the general solution is:

$$Y(t) = C_1 + C_2 t + C_3 e^{2t}$$

So now if we have the IVP with Y(0) = 1, Y'(0) = 0 and Y''(0) = -4 we can find the specific solution by first finding:

$$Y'(t) = C_2 + 2C_3 e^{2t}$$

 $Y''(t) = 4C_3 e^{2t}$

and then solving the system:

$$1 = Y(0) = C_1 + C_3$$

$$0 = Y'(0) = C_2 + 2C_3$$

$$-4 = Y''(0) = 4C_3$$

So that $C_3 = -1$, $C_2 = 2$ and $C_1 = 2$ and the specific solution is:

$$Y(t) = 2 + 2t - e^{2t}$$

5. More about Fundamental Sets:

(a) Natural Fundamental Sets

There's more than just one fundamental set, and one that comes up a lot is called the *natural* fundamental set.

In the second-order case this is the set $\{Y_1, Y_2\}$ with Y_1 having $Y_1(t_I) = 1$ and $Y'_1(t_I) = 0$ and with Y_2 having $Y_2(t_I) = 0$ and $Y'_2(t_I) = 1$.

In the third-order case this is the set $\{Y_1, Y_2, Y_3\}$ with Y_1 having $Y_1(t_I) = 1$, $Y'_1(t_I) = 0$, and $Y''_1(t_I) = 0$, with Y_2 having $Y_2(t_I) = 0$, $Y'_2(t_I) = 1$, and $Y''_2(t_I) = 0$, and with Y_3 having $Y_3(t_I) = 1$, $Y''_3(t_I) = 0$, and $Y''_3(t_I) = 1$,

Beyond there you can probably see the pattern.

(b) Reduction of Order (OMITTED)

The big question of course is where the fundamental set comes from. We'll address that a bit later but for now we have one helper.

If we have one solution $Y_1(t)$ then the second one is very often a multiple of the first. So we can set $Y_2(t) = uY_1(t)$ and when we plug this into the DE and use the fact that $Y_1(t)$ is a solution we end up in a situation where we can find a first-order DE (hence the name) that we can use to find u.

Example: You can check that $Y_1(t) = e^{5t}$ is a solution to y'' - 3y' - 10y = 0. To find the other by reduction of order we put $Y_2(t) = ue^{5t}$. We then find

$$\begin{array}{l} Y_2'(t) = u'e^{5t} + 5ue^{5t} \text{ and} \\ Y_2''(t) = u''e^{5t} + 5u'e^{5t} + 5u'e^{5t} + 25ue^{5t} = u''e^{5t} + 10u'e^{5t} + 25ue^{5t} \end{array}$$

and plug these into the DE:

$$y'' - 3y' - 10y = 0$$

(u''e^{5t} + 10u'e^{5t} + 25ue^{5t}) - 3(u'e^{5t} + 5ue^{5t}) - 10(ue^{5t}) = 0
u'' + 10u' + 25u - 3u' - 15u - 10u = 0
u'' + 7u' = 0

If we let w = u' then this gives us w' + 7w = 0 which has solution $w = Ce^{-7t}$ and so $u' = Ce^{-7t}$ and so $u = -\frac{1}{7}Ce^{-7t} + D$ and another solution is

$$Y_2(t) = \left(-\frac{1}{7}Ce^{-7t} + D\right)e^{5t} = -\frac{1}{7}Ce^{-2t} + De^{5t}$$

Since this is true for any C and D we can pick the solution

$$Y_2(t) = e^{-2t}$$

for which $W[Y_1, Y_2] \neq 0$ and we have our fundamental pair.