MATH 246: Chapter 2 Section 2: Homogeneous Equations - Method and Theory Justin Wyss-Gallifent
Main Topics:

- Definition of Homogeneous.
- Motivational Example for Second Order.
- Theory for Second and Third Order.

1. Introduction: Since even linear higher-order DEs are difficult we are going to simplify even more. For today we're going to look at homogeneous higher-order linear DEs, in which the forcing function $f(t)$ is equal to 0 . That is:

$$
\begin{array}{ll}
\text { First-Order } & y^{\prime}+a(t) y=0 \\
\text { Second-Order } & y^{\prime \prime}+a(t) y^{\prime}+b(t) y=0 \\
\text { Third-Order } & y^{\prime \prime \prime}+a(t) y^{\prime \prime}+b(t) y^{\prime}+c(t) y=0 \\
\vdots & \vdots
\end{array}
$$

2. A Motivational Example: Consider the second-order homogeneous linear DE:

$$
y^{\prime \prime}-y^{\prime}-2 y=0
$$

Next look at the following two functions, don't worry about where they came from:

$$
Y_{1}(t)=e^{2 t} \text { and } Y_{2}(t)=e^{-t}
$$

We can easily see that these are both solutions to the DE by plugging them (and their derivatives) in and checking.
(a) Observation 1 - Getting More Solutions:

Notice that if we take a linear combination of these two, meaning

$$
Y(t)=C_{1} e^{2 t}+C_{2} e^{-t}
$$

where $C_{1}$ and $C_{2}$ are constants. Then we can easily see that this is also a solution to the DE by plugging it (and its derivatives) in and checking.
(b) Observation 2-Getting All Solutions:

We can build new solutions from these two but can we build all solutions this way? Well suppose that we had some solution to the DE, call it $Y(t)$. What we want to know is if we can find $C_{1}$ and $C_{2}$ so that $Y(t)=C_{1} e^{2 t}+C_{2} e^{-t}$ for this $Y(t)$ ?
Well, suppose we find that $Y(0)=y_{0}$ and $Y^{\prime}(0)=y_{1}$. Since $Y^{\prime}(t)=2 C_{1} e^{2 t}-C_{2} e^{-t}$ we would need

$$
\begin{aligned}
y_{0}=Y(0) & =C_{1}+C_{2} \\
y_{1}=Y^{\prime}(0) & =2 C_{2}-C_{2}
\end{aligned}
$$

Can we find such values? Since $\left|\begin{array}{rr}1 & 1 \\ 2 & -1\end{array}\right| \neq 0$ there is a unique solution.
Notice now that since this is a solution to the IVP and since there is only one solution to the IVP this must be the solution we were looking for.
(c) Observation 3 - Anything Special About Those Two?

We can't just start with any two solutions. To see this observe that if we'd started with $Y_{1}(t)=e^{2 t}$ and $Y_{2}(t)=17 e^{2 t}$ that both of these are solutions. Again any linear combination $Y(t)=C_{1} e^{2 t}+C_{2} 17 e^{2 t}$ is a solution. However is every solution to the DE a linear combination? Again, suppose $Y(t)$ is a solution and $Y(0)=y_{0}$ and $Y^{\prime}(0)=y_{1}$. Then $Y^{\prime}(t)=2 C_{1} e^{2 t}+34 C_{2} e^{2 t}$ and we would need

$$
\begin{aligned}
y_{0}=Y(0) & =C_{1}+17 C_{2} \\
y_{1}=Y^{\prime}(0) & =2 C_{1}+34 C_{2}
\end{aligned}
$$

Since $\left|\begin{array}{ll}1 & 17 \\ 2 & 34\end{array}\right|=0$ there may be no solution. That is, we can't guarantee a solution.

## 3. Theory:

(a) Theory for Second-Order Homogeneous: $y^{\prime \prime}+a(t) y^{\prime}+b(t) y=0$

- For a second-order homogeneous linear DE we need to find two solutions $Y_{1}(t)$ and $Y_{2}(t)$ with a special relationship. That relationship is that their Wronskian does not equal the zero function, where:

$$
W\left[Y_{1}, Y_{2}\right]=\left|\begin{array}{cc}
Y_{1} & Y_{2} \\
Y_{1}^{\prime} & Y_{2}^{\prime}
\end{array}\right|
$$

Alternately the two solutions cannot be multiples of each other. They form a fundamental set or fundamental pair of solutions $\left\{Y_{1}(t), Y_{2}(t)\right\}$.

- Every solution is then a linear combination of the fundamental pair. This means the general solution is $Y(t)=C_{1} Y_{1}(t)+C_{2} Y_{2}(t)$.
- A second-order IVP must provide $y\left(t_{I}\right)$ and $y^{\prime}\left(t_{I}\right)$ in order to find the specific solution.
- This solution is unique on the interval of existence and uniqueness which is the largest open interval containing $t_{I}$ on which $a(t)$ and $b(t)$ are differentiable.
(b) Theory for Third-Order Homogeneous: $y^{\prime \prime \prime}+a(t) y^{\prime \prime}+b(t) y^{\prime}+c(t) y=0$
- For a third-order homogeneous linear DE we need to find three solutions $Y_{1}(t), Y_{2}(t)$, and $Y_{3}(t)$ with a special relationship. That relationship is that their Wronskian does not equal the zero function, where:

$$
W\left[Y_{1}, Y_{2}, Y_{3}\right]=\left|\begin{array}{rrr}
Y_{1} & Y_{2} & Y_{3} \\
Y_{1}^{\prime} & Y_{2}^{\prime} & Y_{3}^{\prime} \\
Y_{1}^{\prime \prime} & Y_{2}^{\prime \prime} & Y_{3}^{\prime \prime}
\end{array}\right|
$$

Alternately it must be impossible to write one of the solutions as a linear combination of the others. They form a fundamental set of solutions $\left\{Y_{1}(t), Y_{2}(t), Y_{3}(t)\right\}$.

- Every solution is then a linear combination of the fundamental set. This means the general solution is $Y(t)=C_{1} Y_{1}(t)+C_{2} Y_{2}(t)+C_{3} Y_{3}(t)$.
- A third-order IVP must provide $y\left(t_{I}\right), y^{\prime}\left(t_{I}\right)$, and $y^{\prime \prime}\left(t_{I}\right)$ in order to find the specific solution.
- This solution is unique on the interval of existence and uniqueness which is the largest open interval containing $t_{I}$ on which $a(t)$ and $b(t)$ and $c(t)$ are differentiable.
(c) Theory for Higher-Order:

You can probably see the pattern.
(d) Critical Note: Don't worry about where these fundamental sets are coming from right now, just realize that we (somehow) need obtain them!

## 4. Practice for Both:

Here are some examples:
Example: Consider $y^{\prime \prime}+4 y=0$. First we'll show that $Y_{1}(t)=\sin (2 t)$ and $Y_{2}(t)=$ $\cos (2 t)$ form a fundamental pair. We check they are solutions (omitted) and we check:

$$
W\left[Y_{1}, Y_{2}\right]=\left|\begin{array}{rr}
\sin (2 t) & \cos (2 t) \\
2 \cos (2 t) & -2 \sin (2 t)
\end{array}\right|=-2 \sin ^{2}(2 t)-2 \cos ^{2}(2 t)=-2 \not \equiv 0
$$

This tells us that $Y_{1}(t)$ and $Y_{2}(t)$ form a fundamental pair and that the general solution is:

$$
Y(t)=C_{1} \sin (2 t)+C_{2} \cos (2 t)
$$

So now if we have the IVP with $Y(0)=4$ and $Y^{\prime}(0)=2$ we can find the specific solution by first finding:

$$
Y^{\prime}(t)=2 C_{1} \cos (2 t)-2 C_{2} \sin (2 t)
$$

and then solving the system:

$$
\begin{aligned}
4=Y(0) & =C_{1} \sin (2(0))+C_{2} \cos (2(0))=C_{2} \\
2=Y^{\prime}(0) & =2 C_{1} \cos (2(0))-2 C_{2} \sin (2(0))=2 C_{1}
\end{aligned}
$$

So that $C_{1}=1$ and $C_{2}=4$ and the specific solution is:

$$
Y(t)=\sin (2 t)+4 \cos (2 t)
$$

Example: Consider $\left(1+t^{2}\right) y^{\prime \prime}-2 t y^{\prime}+2 y=0$. First we'll show that $Y_{1}(t)=t$ and $Y_{2}(t)=t^{2}-1$ form a fundamental pair. Notice that it doesn't matter whether we divide by $1+t^{2}$ or not when we check these. We check they are solutions (omitted) and we check:

$$
W\left[Y_{1}, Y_{2}\right]=\left|\begin{array}{rr}
t & t^{2}-1 \\
1 & 2 t
\end{array}\right|=2 t^{2}-\left(t^{2}-1\right)=t^{2}+1 \not \equiv 0
$$

This tells us that $Y_{1}(t)$ and $Y_{2}(t)$ form a fundamental pair and that the general solution is:

$$
Y(t)=C_{1} t+C_{2}\left(t^{2}-1\right)
$$

So now if we have the IVP with $Y(2)=-5$ and $Y^{\prime}(2)=7$ we can find the specific solution by first finding:

$$
Y^{\prime}(t)=C_{1}+2 C_{2} t
$$

and then solving the system:

$$
\begin{aligned}
-5=Y(2) & =C_{1}(2)+C_{2}\left(2^{2}-1\right)=2 C_{1}+3 C_{2} \\
7=Y^{\prime}(2) & =C_{1}+2 C_{2}(2)=C_{1}+4 C_{2}
\end{aligned}
$$

So that $C_{1}=-\frac{41}{5}$ and $C_{2}=-\frac{19}{5}$ and the specific solution is:

$$
Y(t)=-\frac{41}{5} t+\frac{19}{5}\left(t^{2}-1\right)
$$

Example: Consider $D^{3} y-2 D^{2} y=0$ First we'll show that $Y_{1}(t)=1, Y_{2}(t)=t$ and $Y_{3}(t)=e^{2 t}$ form a fundamental set. We check they are solutions (omitted) and we check:

$$
W\left[Y_{1}, Y_{2}, Y_{3}\right]=\left|\begin{array}{ccc}
1 & t & e^{2 t} \\
0 & 1 & 2 e^{2 t} \\
0 & 0 & 4 e^{2 t}
\end{array}\right|=4 e^{2 t} \not \equiv 0
$$

This tells us that $Y_{1}(t), Y_{2}(t)$ and $Y_{3}(t)$ form a fundamental set and that the general solution is:

$$
Y(t)=C_{1}+C_{2} t+C_{3} e^{2 t}
$$

So now if we have the IVP with $Y(0)=1, Y^{\prime}(0)=0$ and $Y^{\prime \prime}(0)=-4$ we can find the specific solution by first finding:

$$
\begin{aligned}
Y^{\prime}(t) & =C_{2}+2 C_{3} e^{2 t} \\
Y^{\prime \prime}(t) & =4 C_{3} e^{2 t}
\end{aligned}
$$

and then solving the system:

$$
\begin{aligned}
1=Y(0) & =C_{1}+C_{3} \\
0=Y^{\prime}(0) & =C_{2}+2 C_{3} \\
-4=Y^{\prime \prime}(0) & =4 C_{3}
\end{aligned}
$$

So that $C_{3}=-1, C_{2}=2$ and $C_{1}=2$ and the specific solution is:

$$
Y(t)=2+2 t-e^{2 t}
$$

## 5. More about Fundamental Sets:

## (a) Natural Fundamental Sets

There's more than just one fundamental set, and one that comes up a lot is called the natural fundamental set.
In the second-order case this is the set $\left\{Y_{1}, Y_{2}\right\}$ with $Y_{1}$ having $Y_{1}\left(t_{I}\right)=1$ and $Y_{1}^{\prime}\left(t_{I}\right)=0$ and with $Y_{2}$ having $Y_{2}\left(t_{I}\right)=0$ and $Y_{2}^{\prime}\left(t_{I}\right)=1$.
In the third-order case this is the set $\left\{Y_{1}, Y_{2}, Y_{3}\right\}$ with $Y_{1}$ having $Y_{1}\left(t_{I}\right)=1, Y_{1}^{\prime}\left(t_{I}\right)=0$, and $Y_{1}^{\prime \prime}\left(t_{I}\right)=0$, with $Y_{2}$ having $Y_{2}\left(t_{I}\right)=0, Y_{2}^{\prime}\left(t_{I}\right)=1$, and $Y_{2}^{\prime \prime}\left(t_{I}\right)=0$, and with $Y_{3}$ having $Y_{3}\left(t_{I}\right)=1, Y_{3}^{\prime}\left(t_{I}\right)=0$, and $Y_{3}^{\prime \prime}\left(t_{I}\right)=1$,
Beyond there you can probably see the pattern.
(b) Reduction of Order (OMITTED)

The big question of course is where the fundamental set comes from. We'll address that a bit later but for now we have one helper.
If we have one solution $Y_{1}(t)$ then the second one is very often a multiple of the first. So we can set $Y_{2}(t)=u Y_{1}(t)$ and when we plug this into the DE and use the fact that $Y_{1}(t)$ is a solution we end up in a situation where we can find a first-order DE (hence the name) that we can use to find $u$.
Example: You can check that $Y_{1}(t)=e^{5 t}$ is a solution to $y^{\prime \prime}-3 y^{\prime}-10 y=0$. To find the other by reduction of order we put $Y_{2}(t)=u e^{5 t}$. We then find

$$
\begin{gathered}
Y_{2}^{\prime}(t)=u^{\prime} e^{5 t}+5 u e^{5 t} \text { and } \\
Y_{2}^{\prime \prime}(t)=u^{\prime \prime} e^{5 t}+5 u^{\prime} e^{5 t}+5 u^{\prime} e^{5 t}+25 u e^{5 t}=u^{\prime \prime} e^{5 t}+10 u^{\prime} e^{5 t}+25 u e^{5 t}
\end{gathered}
$$

and plug these into the DE:

$$
\begin{aligned}
y^{\prime \prime}-3 y^{\prime}-10 y & =0 \\
\left(u^{\prime \prime} e^{5 t}+10 u^{\prime} e^{5 t}+25 u e^{5 t}\right)-3\left(u^{\prime} e^{5 t}+5 u e^{5 t}\right)-10\left(u e^{5 t}\right) & =0 \\
u^{\prime \prime}+10 u^{\prime}+25 u-3 u^{\prime}-15 u-10 u & =0 \\
u^{\prime \prime}+7 u^{\prime} & =0
\end{aligned}
$$

If we let $w=u^{\prime}$ then this gives us $w^{\prime}+7 w=0$ which has solution $w=C e^{-7 t}$ and so $u^{\prime}=C e^{-7 t}$ and so $u=-\frac{1}{7} C e^{-7 t}+D$ and another solution is

$$
Y_{2}(t)=\left(-\frac{1}{7} C e^{-7 t}+D\right) e^{5 t}=-\frac{1}{7} C e^{-2 t}+D e^{5 t}
$$

Since this is true for any $C$ and $D$ we can pick the solution

$$
Y_{2}(t)=e^{-2 t}
$$

for which $W\left[Y_{1}, Y_{2}\right] \not \equiv 0$ and we have our fundamental pair.

