MATH 246: Chapter 3 Section 10: Population Dynamics
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Main Topics:

- Predator-Prey Models
- Competing Species Models
- Cooperating Species Models

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1. **Predator-Prey Models**

   Consider a interaction between predators and prey. Suppose the number of prey is $x(t)$ while the number of predators is $y(t)$. A simple but reasonable system of differential equations modeling these could be:

$$x' = (r - ax - by)x$$
$$y' = (-s + cx - dy)y$$

   To understand the meaning of these constants, consider that $r - ax - by$ is the growth rate for prey while $-x + cx - dy$ is the growth rate for predators. This is the reason they're multiplied by x and y respectively to get x' and y'. Moreover:

   - The constant $r > 0$ gives the intrinsic growth rate of the prey. This is positive because by default (in absence of predators) the prey will reproduce.
   - The growth rate of prey may decline as the number of prey grows due to competetiveness. This is managed by the constant $a \geq 0$.
   - The growth rate of the prety will decline as the number of predators grows. This is managed by the constant $d > 0$.
   - The constant $s > 0$ gives the intrinsic growth rate of the predators. We have $-s$ because by default (in absense of prey) the predators will die out.
   - The growth rate of predators will increase with the number of prey. This is managed by the constant $c > 0$.
   - The growth rate of predators may decline as the number of predators grows due to competetiveness. This is managed by the constant $d \geq 0$.

   Our goal will be to analyze such systems and understand what happens to the populations in the long term.

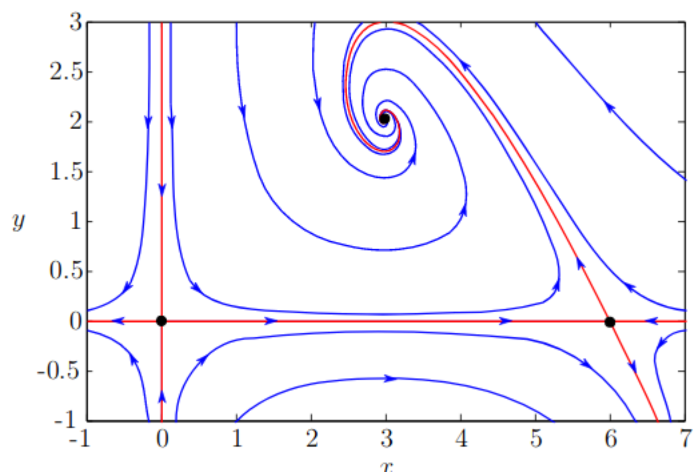
Example: Consider the model:

$$x' = (12 - 2x - 3y)x$$
$$y' = (-15 + 5x)y$$

There are three stationary points which we analyze as follows:

- $(0,0)$ has $\partial\bar{F} = \begin{bmatrix} 12 & 0 \\ 0 & -15 \end{bmatrix}$ with eigenpairs $\left(12, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$, $\left(-15, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$.
  It follows that this is a saddle.
- $(6,0)$ has $\partial\bar{F} = \begin{bmatrix} -12 & -18 \\ 0 & 15 \end{bmatrix}$ with eigenpairs $\left(-12, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$, $\left(15, \begin{bmatrix} -2 \\ 3 \end{bmatrix}\right)$
  It follows that this is a saddle.
- $(3,2)$ has $\partial\bar{F} = \begin{bmatrix} -6 & -9 \\ 10 & 0 \end{bmatrix}$ with eigenvalues $-3 \pm 9i$.
  It follows that this is a counterclockwise spiral sink.

The following picture was pilfered from Levermore's notes:



So now an initial population of $(0.1, 2)$ will undergo a decrease in predators, resulting in an increase in prey, resulting in an increase in predators, resulting in a decrease in prey, and so on, and will eventually spiral into the stable point $(3, 2)$.

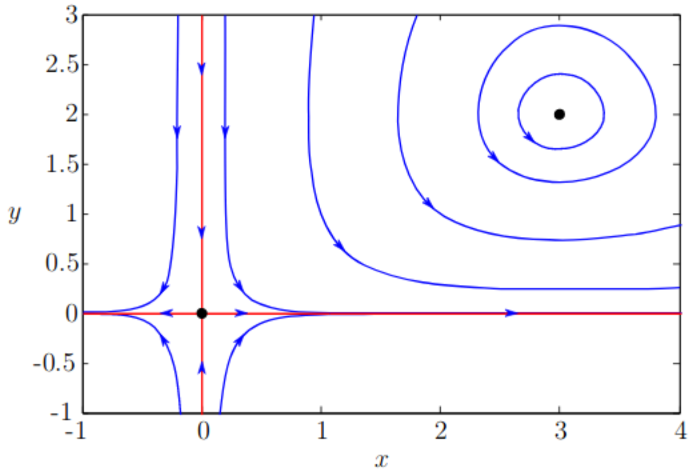
Example: Consider the model:

$$x' = (6 - 3y)x$$
$$y' = (-15 + 5x)y$$

There are two stationary points which we analyze as follows:

- $(0,0)$ has $\partial \bar{F} = \begin{bmatrix} 12 & 0 \\ 0 & -15 \end{bmatrix}$ with eigenpairs $\left(6, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right), \left(-15, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$.
  It follows that this is a saddle.
- $(3,2)$ has $\partial \bar{F} = \begin{bmatrix} 0 & -9 \\ 10 & 0 \end{bmatrix}$ with eigenvalues $0 \pm 90i$.
  It follows that this is a counterclockwise circle.

The following picture was pilfered from Levermore's notes:



So now an initial population in the first quadrant will tend to circle around $(3,2)$ but it will not approach it in a spiral sense.

2. **Competing Species Models**

Competing species models look liks this:

$$x' = (r - ax - by)x$$
$$y' = (s - cx - dy)y$$

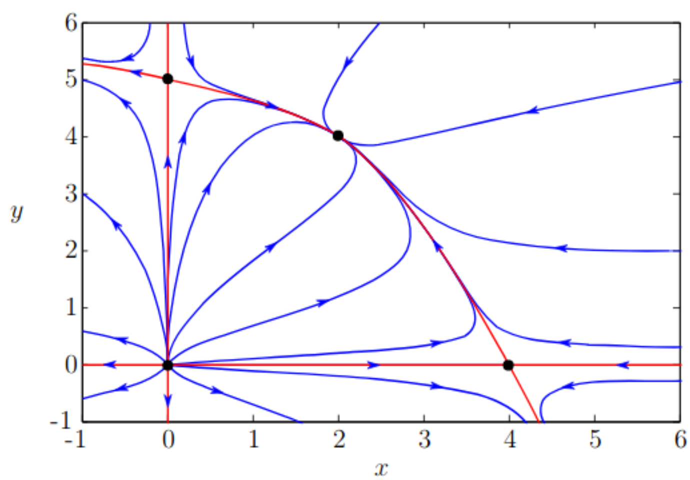
Example: Consider the model:

$$x' = (16 - 4x - 2y)x$$
$$y' = (10 - x - 2y)y$$

There are four stationary points which we analyze as follows:

- $(0,0)$ has $\partial\bar{F} = \begin{bmatrix} 16 & 0 \\ 0 & 10 \end{bmatrix}$ with eigenpairs $\left(16, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right), \left(10, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$.
  It follows that this is a nodal source..
- $(0,5)$ has $\partial\bar{F} = \begin{bmatrix} 6 & 0 \\ -5 & -10 \end{bmatrix}$ with eigenpairs $\left(6, \begin{bmatrix} 16 \\ -5 \end{bmatrix}\right), \left(-10, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$
  It follows that this is a saddle.
- $(4,0)$ has $\partial\bar{F} = \begin{bmatrix} -16 & -8 \\ 0 & 6 \end{bmatrix}$ with eigenpairs $\left(-16, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right), \left(6, \begin{bmatrix} -4 \\ 11 \end{bmatrix}\right)$
  It follows that this is a saddle.
- $(2,4)$ has $\partial\bar{F} = \begin{bmatrix} -8 & -4 \\ -4 & -8 \end{bmatrix}$ with eigenpairs $\left(-12, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right), \left(-4, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$
  It follows that this is a nodal sink.

The following picture was pilfered from Levermore's notes:



We see that if an initial population has both x and y positive then it will tend towards $(2,4)$ but this can happen in a variety of ways.
For example if we start at $(10,0.1)$ this means there are lots of species x and few of species y. Because there are a lot of species x they are constrained by resources and hence their population drops. When it gets close to $x = 4$ however the rate of drop decreases and at that point resources are not so constraining it tends to start to stabilize. However at that point since x and y are competing y can grow now, since there aren't so many x. And so it does, and this causes x to drop more. In the long term (infinity) the pair heads to $(2,4)$.

3. **Cooperating Species Models**

Cooperating species models look like this;

$$x' = (r - ax + by)x$$
$$y' = (s + cx - dy)y$$

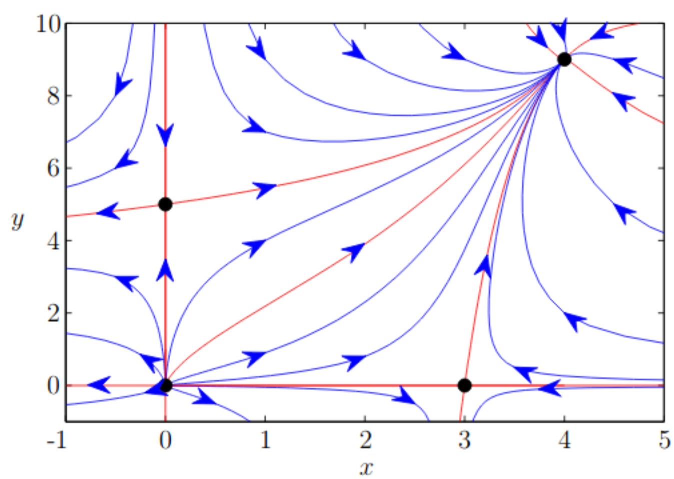
Example: Consider the model:

$$x' = (27 - 9x + y)x$$
$$y' = (20 + 4x - 4y)y$$

There are four stationary points which we analyze as follows:

- $(0,0)$ has $\partial \bar{F} = \begin{bmatrix} 27 & 0 \\ 0 & 20 \end{bmatrix}$ with eigenpairs $\left(27, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right), \left(20, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$.
  It follows that this is a nodal source..
- $(0,5)$ has $\partial \bar{F} = \begin{bmatrix} 32 & 0 \\ 20 & -20 \end{bmatrix}$ with eigenpairs $\left(32, \begin{bmatrix} 13 \\ 5 \end{bmatrix}\right), \left(-20, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$
  It follows that this is a saddle.
- $(3,0)$ has $\partial \bar{F} = \begin{bmatrix} -27 & 3 \\ 0 & 32 \end{bmatrix}$ with eigenpairs $\left(-27, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right), \left(32, \begin{bmatrix} 3 \\ 59 \end{bmatrix}\right)$
  It follows that this is a saddle.
- $(4,9)$ has $\partial \bar{F} = \begin{bmatrix} -36 & 9 \\ 16 & -36 \end{bmatrix}$ with eigenpairs $\left(-24, \begin{bmatrix} 3 \\ 4 \end{bmatrix}\right), \left(-48, \begin{bmatrix} 3 \\ -4 \end{bmatrix}\right)$
  It follows that this is a nodal sink.

The following picture was pilfered from Levermore's notes:



We see that if an initial population has both x and y positive then it will tend towards $(4,9)$ but this can happen in a variety of ways.