MATH 246: Chapter 3 Section 3: Matrix and Vector Essentials
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Main Topics:

- Basic Definitions
- Combining Matrices and Vectors
- Inverses
- Determinants
- Transpose and Complex Conjugate
- Eigenstuff

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1. **Introduction:**

   It's far easier to manage systems of differential equations when we can rephrase them in the language of matrices and vectors. To that end, here are the essentials.

2. **Basic Definitions:**

   (a) A *matrix* is a rectangular array of numbers. It has size $m \times n$ if there are m rows and n columns. Matrices are typically denoted using capital letters:

   Example: Here is a 3×4 matrix:

   $$A = \begin{bmatrix} 1 & 3 & -1 & 0 \\ 0 & 5 & 0 & 17 \\ 2 & 2 & -7 & 3 \end{bmatrix}$$

   (b) Most of the matrices in this class will be *square*, meaning they have the same number of rows and columns. Mostly we'll deal with 2×2 and 3×3 matrices.

   (c) The *identity matrix* I_n is the square $n \times n$ matrix with 1s on the *main diagonal* (upper-left to lower right) and 0s elsewhere. When the size is clear from context we just write I.

   Example:

   $$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

   (d) The *zero matrix* is matrix of all zeros.

   (e) A *vector* is a matrix which is a single column. Vectors are usually denoted in lower-case with a bar over the letter.

   Example: $\bar{a} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$

3. **Combining Matrices and Vectors:**

(a) We can add matrices and vectors by adding matching entries provided they both have the same size.

Example: For example:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 6 & 2 \end{bmatrix} = \begin{bmatrix} 1-1 & 2+0 \\ 3+6 & 4+2 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 9 & 8 \end{bmatrix}$$

(b) We can multiply an $n \times n$ matrix A by a vector $\bar{x}$ with n entries to get a new vector with n entries. The formal definition of this is that we take the linear combination of the columns of A using the weights in $\bar{x}$. More informally we do this by multiplying each row of the matrix by the vector (element by element and add). This is easier to see:

Example: We have:

$$\begin{aligned} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 4 & 7 \\ 8 & -1 & 5 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix} &= 5 \begin{bmatrix} 1 \\ 0 \\ 8 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix} + 2 \begin{bmatrix} -3 \\ 7 \\ 5 \end{bmatrix} \\ &= \begin{bmatrix} (1)(5) + (2)(3) + (-3)(2) \\ (0)(5) + (4)(3) + (7)(2) \\ (8)(5) + (-1)(3) + (5)(2) \end{bmatrix} \\ &= \begin{bmatrix} 5 \\ 26 \\ 47 \end{bmatrix} \end{aligned}$$

(c) We can multiply an $n \times n$ matrix by another $n \times n$ matrix by multiplying the first matrix by each of the columns in the second matrix as if it were just a vector, then taking these new vectors an putting them together in a new matrix.

Example: Here it is with lots of brackets to help you see what's going on:

$$\begin{aligned} \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 5 & 9 \end{bmatrix} &= \left[\begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 5 \end{bmatrix} \quad \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 9 \end{bmatrix} \right] \\ &= \left[\begin{bmatrix} (2)(-1) + (1)(5) \\ (4)(-1) + (3)(5) \end{bmatrix} \begin{bmatrix} (2)(3) + (1)(9) \\ (4)(3) + (3)(9) \end{bmatrix} \right] \\ &= \begin{bmatrix} 3 & 15 \\ 9 & 39 \end{bmatrix} \end{aligned}$$

(d) It's almost always the case that for matrices A and B that $AB \neq BA$.

(e) The identity matrix acts like the number 1 in that for any matrix A we have:

$$AI = IA = A$$

4. **Determinants:**

   (a) The *determinant* of a matrix, denoted $\det A$ or by putting the matrix in vertical bars instead of brackets, is a number calculated from the matrix. We've seen this for 2×2 and 3×3 matrices.

   (b) Properties include:

   - A matrix A has an inverse if and only if $\det A \neq 0$.
   - For a 2×2 case $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

5. **Inverses:**

   (a) The *inverse* of an $n \times n$ matrix A is another matrix denoted A^{-1} such that $AA^{-1} = A^{-1}A = I$. It's like a "reciprocal" for matrices.

   (b) For the 2×2 size there is a formula:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example: For example:

$$\begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix}^{-1} = \frac{1}{(1)(-2) - (3)(2)} \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix} = \frac{1}{-8} \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -1/4 & 3/8 \\ -1/8 & -1/4 \end{bmatrix}$$

   (c) Properties include:

   - $\left(A^{-1}\right)^{-1} = A$
   - $(AB)^{-1} = B^{-1}A^{-1}$
   - $(\alpha A)^{-1} = \frac{1}{\alpha} A^{-1}$
   - We have $\det\left(A^{-1}\right) = 1/(\det A)$.

6. **Transpose and Complex Conjugate:**

   (a) The *transpose* of an $n \times n$ matrix A, denoted A^T, is found by reflecting the matrix in its main diagonal.

Example: We have:

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 4 & 7 \\ 8 & -1 & 5 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 8 \\ 2 & 4 & -1 \\ -3 & 7 & 5 \end{bmatrix}$$

   (b) A matrix may have complex numbers in it, in which case its *comples conjugate* denoted either $\bar{A}$ or A^c, is found by taking the complex conjugate of each entry.

Example: We have:

$$\begin{bmatrix} 1 - 2i & 5 \\ 5 + i & 7 + 8i \end{bmatrix}^C = \begin{bmatrix} 1 + 2i & 5 \\ 5 - i & 7 - 8i \end{bmatrix}$$

7. **Eigenstuff:**

If we have a matrix, the determinant is the most important number associated to it. After the determinant the next most important items are eigenvalues and eigenvectors.

(a) If A is an $n \times n$ matrix, an *eigenvalue* of A is a number λ with the property that there is some $\bar{v} \neq \bar{0}$ such that $A\bar{v} = \lambda\bar{v}$. The vector $\bar{v}$ is then an *eigenvector* associated to λ and we say that $(\lambda, \bar{v})$ is an *eigenpair* of A.

Example: Observe that:

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

so we would say that $\lambda = 3$ is an eigenvalue, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector and $\left(3, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ is an eigenpair for the matrix $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

(b) Any nonzero multiple of an eigenvector is also an eigenvector, so in the above example $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 17 \\ 17 \end{bmatrix}$ and $\begin{bmatrix} -7 \\ -7 \end{bmatrix}$ are all eigenvectors for the same eigenvalue.

(c) If we have a complex eigenpair $(\lambda, \bar{v})$ then $(\bar{\lambda}, \bar{\bar{v}})$ is also an eigenpair.

(d) Given an $n \times n$ matrix A, the value λ will be an eigenvalue if and only if there is some $\bar{v} \neq \bar{0}$ such that

$$A\bar{v} = \lambda\bar{v}$$

We can manipulate this:

$$\begin{aligned} \lambda\bar{v} - A\bar{v} &= \bar{0} \\ \lambda I\bar{v} - A\bar{v} &= \bar{0} \\ (\lambda I - A)\,\bar{v} &= \bar{0} \end{aligned}$$

This will have a nontrivial solution precisely when:

$$\det(\lambda I - A) = 0$$

So what we do is we define the characteristic polynomial of A as:

$$p(z) = \det(zI - A)$$

Then we know that the eigenvalues of A are the roots of this characteristic polynomial.

Example: To find the eigenvalues of

$$A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$$

we find

$$\begin{aligned} p(z) &= \det(zI - A) \\ &= \det\left(z\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}\right) \\ &= \det\left(\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}\right) \\ &= \det\begin{bmatrix} z-3 & -2 \\ -2 & z-3 \end{bmatrix} \\ &= (z-3)(z-3) - 4 \\ &= z^2 - 6z + 5 \\ &= (z-5)(z-1) \end{aligned}$$

The eigenvalues are then the roots so $\lambda_1 = 5$ and $\lambda_2 = 1$.

Example: To find the eigenvalues of

$$A = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}$$

we find

$$\begin{aligned} p(z) &= \det(zI - A) \\ &= \det\left(z\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}\right) \\ &= \det\left(\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}\right) \\ &= \det\begin{bmatrix} z-4 & 1 \\ -1 & z-2 \end{bmatrix} \\ &= (z-4)(z-2) + 1 \\ &= z^2 - 6z + 9 \\ &= (z-3)^2 \end{aligned}$$

The only eigenvalue is the root $\lambda = 3$. However this multiplicity counts, so we can think $\lambda_1 = 3$ and $\lambda_2 = 3$.

Example: To find the eigenvalues of

$$A = \begin{bmatrix} 3 & 2 \\ -2 & 3 \end{bmatrix}$$

we find

$$\begin{aligned} p(z) &= \det(zI - A) \\ &= \det\left(z \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ -2 & 3 \end{bmatrix} \right) \\ &= \det\left(\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ -2 & 3 \end{bmatrix} \right) \\ &= \det \begin{bmatrix} z-3 & -2 \\ 2 & z-3 \end{bmatrix} \\ &= (z-3)(z-3) + 4 \\ &= z^2 - 6z + 13 \end{aligned}$$

This does not factor so we use the quadratic formula:

$$z = \frac{6 \pm \sqrt{(-6)^2 - 4(1)(13)}}{2} = 3 \pm 2i$$

The eigenvalues are then $\lambda_1 = 3 + 2i$ and $\lambda_2 = 3 - 2i$.

(e) Once we find the eigenvalues we take each eigenvalue $z = \lambda$ and solve the matrix equation $A\bar{v} = \lambda\bar{v}$, or $A\bar{v} - \lambda\bar{v} = \bar{0}$, or $(A - \lambda I)\bar{v} = \bar{0}$. This can be fairly intensive for large cases. For the 2×2 case there is a trick, though, which is really useful:

For λ_1 choose any nonzero column of $A - \lambda_2 I$.
For λ_2 choose any nonzero column of $A - \lambda_1 I$.

Example: We saw that the eigenvalues for $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$ are $\lambda_1 = 5$ and $\lambda_2 = 1$. Then:

- For $\lambda_1 = 5$ choose any nonzero column of
$$A - \lambda_2 I = A - 1I = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$
so $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ will do. Since any multiple of this works, pick the nicer $\bar{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
- For $\lambda_2 = 1$ choose any nonzero column of
$$A - \lambda_1 I = A - 5I = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}$$
so $\begin{bmatrix} -2 \\ 2 \end{bmatrix}$ will do. Since any multiple of this works, pick the nicer $\bar{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

We thus have eigenpairs $\left(5, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ and $\left(1, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$.

Example: We saw that the eigenvalue for $A = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}$ is $\lambda_1 = \lambda_2 = 3$. Then:

- For $\lambda_1 = 3$ choose any nonzero column of $A - \lambda_2 I = A - 3I = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ so $\bar{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ will do.
- Notice that $\lambda_2 = \lambda_1$ so we get nothing new.

We thus have the single eigenpair $\left(3, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$.

Example: We saw that the eigenvalues for $A = \begin{bmatrix} 3 & 2 \\ -2 & 3 \end{bmatrix}$ are $\lambda_1 = 3 + 2i$ and $\lambda_2 = 3 - 2i$. Then:

- For $\lambda_1 = 3 + 2i$ choose any nonzero column of
$$A - \lambda_2 I = A - (3 + 2i)I = \begin{bmatrix} 3 & 2 \\ -2 & 3 \end{bmatrix} - \begin{bmatrix} 3 + 2i & 0 \\ 0 & 3 + 2i \end{bmatrix} = \begin{bmatrix} 2i & 2 \\ -2 & -2i \end{bmatrix}$$
so $\begin{bmatrix} 2i \\ -2 \end{bmatrix}$ will do. Since any multiple of this works, we pick the nicer $\bar{v}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$.
- We know from earlier that for $\lambda_2 = 3 - 2i$ we can use the conjugate so $\bar{v}_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$.

We thus have eigenpairs $\left(3 + 2i, \begin{bmatrix} 1 \\ i \end{bmatrix}\right)$ and $\left(3 - 2i, \begin{bmatrix} 1 \\ -i \end{bmatrix}\right)$.