MATH 246: Chapter 0 Section 0
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1. What is a differential equation and what does it mean to solve one?

   (a) The most straightforward definition of a *differential equation* (a DE) is that it's an equation involving some or all of the following: An unknown function, derivatives of that function, and other functions of the same variable(s).

   Example: $f'(t) + f(t) = 10$ in which f is our unknown function of t.
   Example: $y'' + 3y' - xy = 6$ in which y is our unknown function of x.
   Example: $t^2 f''(t) = 5 - f'(t)(\sin t)$ in which f is our unknown function of t.
   Example: $17\frac{dy}{dx} - x\frac{d^2y}{dx^2} = xy$ in which y is our unknown function of x.
   Example: $\partial_x u + sin(x)\partial_y u = y^3 \partial_{xy} u$ in which u is our unknown function of both x and y.

   (b) *Solving a DE* means finding a function which makes the DE true when you plug that function in. For the following don't worry about where the solutions came from, just notice that they work.

   Example: $f(t) = e^t$ is a solution to the DE $f(t) - f'(t) = 0$.
   Example: $f(t) = \sin(t)$ is a solution to the DE $f(t) + f''(t) = 0$.
   Example: $f(t) = t + e^{2t}$ is a solution to the DE $f''(t) + 4t = 4f(t)$.

   Just as regular equations can have more than one solution ($x^2 - 9 = 0$ has two) so can a DE. In fact usually a DE will have infinitely many solutions.

   Example: $f(t) = 487e^t$ is another solution to the DE $f(t) - f'(t) = 0$. You can probably see lots more now.

2. Associated definitions

   (a) A DE is called *ordinary* (so an ODE) if the unknown function is just a function of one variable. Otherwise it's *partial* (so a PDE).

   Example: $f'(t) + 3tf''(t) = e^t$ is an ODE.
   Example: $u_x(x, y) + u_{yx}(x, y) + y = 3$ is a PDE.

   (b) The *order* of a DE is the highest derivative that appears in it. We say things like *first-order* and *second-order* and so on.

   Example: $x^7 f'(x) + (\cos x)f(x) + x = e^x$ is first-order.
   Example: $tf(t) + e^t f''(t) = 1 - f'(t)$ is second-order.

   (c) A DE is *linear* if it can be written as a sum/difference of some or all of:

   - An unknown f multiplied by a coefficient.
   - Derivatives of the unknown f multiplied by coefficients.
   - Coefficients.

   By *coefficients* we mean they can be other functions of the same variables, including just constants, including 0.

   Example: The DE $5tf(t) + (\ln t)f'(t) = 5$ is linear.
   Example: The DE $(\tan t)f(t) + t^3 f'(t) + 7f''(t) = 1$ is linear.
   Example: The DE $f(t)\sqrt{t} + (1 - t)f'''(t) = f'(t)$ is linear.
   Example: The DE $f(t)^2 + f'(t) = 7$ is nonlinear because the $f(t)^2$ is not permitted.
   Example: The DE $\sin(y') + y' + y = x$ is nonlinear because the $\sin(y')$ is not permitted.
   Example: The DE $y'y + y = xy$ is nonlinear because the $y'y$ is not permitted.

Special Cases: A *first-order linear ordinary differential equation* using the variable t and the unknown function y will have the form

$$a(t)y' + b(t)y = c(t)$$

A *second-order linear ordinary differential equation* using the variable t will have the form

$$a(t)y'' + b(t)y' + c(t)y = d(t)$$

And so on...

(d) A *system* of DEs is just that, a collection of more than one DE where the goal is to find a single function that makes them all true. The *order* of such a system is the highest derivative that appears in any of the DEs.

Example: A first-order system of two linear ODEs:

$$\begin{aligned} ty + t^2y' &= e^t \\ 3y + 5y' &= \sin(t) \end{aligned}$$

3. Moving onwards.

At this point you can probably start to wrap your head around which DEs looks like they might be easier to handle. The following is a list of DEs of increasing complexity. Even though you don't really know how to solve any of these just yet (that's not true, you can do the first one!) you can almost certainly look at them in order and get an apprection for the fact that they start pretty nice and get more convoluted! Don't worry that some of the words on the right might not make sense.

$y' = t^2$	Explicit first order linear ODE
$5y' - 4y = 0$	Homogeneous first order linear ODE with constant coefficients
$2y'' + 5y' - 4y = 0$	Homogeneous second order linear ODE with constant coefficients
$7y' - 2y = t$	Nonhomogeneous first order linear ODE with constant coefficients
$t^2y' + e^ty = 1 + t$	Nonhomogeneous first order linear ODE

With some quirky exceptions our approach will pretty much be like that in that we'll first tackle the easier types. This will help us develop some theory which will then support us as we move to the more complicated types, and then to systems of these.