MATH 246: Chapter 1 Section 1
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1. Introductory overview of first-order ODEs.

   (a) A first-order ODE is permitted to have an unknown function y (of a single variable, say t) its derivative y' and then some other functions of t.

   Example: $t(y')^2 + y = \sin t$

   Example: $y' - ty = e^{2t}$

   Example: $\sin(y') + e^{y'} = t$

   (b) In general these can be very hard! For now let's restrict ourselves a bit to first-order ODEs that have the form $y' = f(t,y)$ because in first-order ODE getting to this point is usually algebra.

   That f is confusing, it's not the unknown function but rather it just represents the fact that we can have a bunch of y and t on the right hand side. In other words things like this:

   Example: $y' = ty$

   Example: $y' = 4t - 8y$

   Example: $y' = \frac{y}{t}$.

2. Explicit first-order DEs.

3. (a) Because solving even first-order ODEs is hard we'll go down even further and look at *explicit* first-order ODEs that have the form $y' = f(t)$.

   Example: $y' = t^2$.

   Example: $y' = 4t + \sin t$.

   (b) At this point you might have an epiphany and realize that often you can solve these because solving these is as easy as integrating the right side.

   Example: $y' = t^2$. To solve this we integrate to get $y = \frac{1}{3}t^2 + C$ for any constant C.

   Example: $y' = 4t + \sin t$. To solve this we integrate to get $y = 2t^2 - \cos t + C$ for any constant C.

4. General solutions, initial value problems, specific (particular) solutions

   (a) We've started to notice that we can have many solutions to a DE. In the explicit DEs above get a constant C which can be anything.

   (b) A *general solution* to a DE is a solution involving constants and for which different constants will give all solutions.

   (c) A *specific solution* or a *particular solution* is a solution in which a specific (particular) choice of constant(s) has been made.

   Example: The general solution to $y' = t^2$ is $y = \frac{1}{3}t^3 + C$. Some specific solutions are $y = \frac{1}{3}t + 1$, $y = \frac{1}{3}t - 107$ and $y = \frac{1}{3}t + \pi$.

   (d) Often when we encounter a DE it comes pre-packaged with an *initial value*, or IV. In our simple exact case (and in many future cases) this will be an insistance that $y(t_I) = y_I$ for specific t_I and y_I. The DE and the IV together form an *initial value problem* or IVP. It's very common that $t_I = 0$ but this isn't always the case!

Example: $y' = 2t$ with $y(0) = 3$ is an IVP.

Example: $y' = 2t$ with $y(0) = 5$ is an IVP with the same DE but different IV.

Example: $y' = 2t$ with $y(1) = 3$ is an IVP with again the same DE but different IV.

(e) When we're given an IVP the idea will be to first solve the DE to get the general solution and then use the IV to get the specific solution.

Example: $y' = 2t$ with $y(0) = 3$. First we find $y = t^2 + C$, the general solution, and then $y(0) = 0^2 + C = 3$ so $C = 3$ and the specific solution is $y = t^2 + 3$.

Example: $y' = 2t$ with $y(1) = 3$. First we find $y = t^2 + C$, the general solution, and then $y(1) = 1^2 + C = 3$ so $C = 2$ and the specific solution is $y = t^2 + 2$.

5. Some Theory which will arise again and again.

(a) Finding a solution to $y' = f(t, y)$; we will be a little more specific. We will not just say that y is a solution but that y is a solution on an interval (a, b). This will mean:

i. The derivative y' (derived from y) is defined for every t in (a, b).

ii. The right side, $f(t, y)$, is defined for every t in (a, b) when y is plugged in.

iii. The DE is true for every t in (a, b).

Example: Consider the DE $y' = -\frac{t}{y}$. We claim that $y = \sqrt{1 - t^2}$ is a solution on $(-1, 1)$. Checking the three things above:

i. $y = \sqrt{1 - t^2}$ so $y' = -\frac{t}{\sqrt{1-t^2}}$ which is defined for every t in $(-1, 1)$.

ii. The right side is $-\frac{t}{y}$ which equals $-\frac{t}{\sqrt{1-t^2}}$ which is defined for every t in $(-1, 1)$.

iii. The DE holds true for every t in $(0, \infty)$.

Comment: This y is not a solution on $[-1, 1]$ because even though y itself is defined at $t = \pm 1$ we see that y' is not!

(b) Intervals of Existence and Theorey for Explicit IVPs:

We now know that solving the explicit ODE given by $y' = f(t)$ is as easy (or hard) as integrating $f(t)$. However the Fundamental Theorem of Calculus tells us something interesting. It states that if a function is continuous then it is integrable. This means that even if we can't actually integrate $f(t)$ using techniques that we know, we still know there is a solution. Moreover that solution will exist on an interval where $f(t)$ is continuous.

The practical upshot of this is that when we're solving an IVP with $y' = f(t)$ and $y(t_I) = y_I$, there will be a solution on the largest interval (a, b) which contains t_I and on which $f(t)$ is continuous. This interval is called the *interval of existence* of the solution and holds whether or not we can actually, in practice, find that solution.

Example: $y' = \frac{1}{t^2}$ with $y(1) = 5$. We notice the largest open interval containing $t_I = 1$ on which $\frac{1}{t^2}$ is defined is $(0, \infty)$ and so this is the IE. Notice that we don't need to solve it, but we could, since the general solution is $y = -\frac{1}{t} + C$ and then $y(1) = -1 + C = 5$ so $C = 4$ and the specific solution is $y = -\frac{1}{t} + 4$.

Example: $y' = \frac{t}{(t-3)(t+6)}$ with $t(0) = 17$. We notice the largest open interval containing $t_I = 0$ on which $\frac{t}{(t-3)(t+6)}$ is defined is $(-6, 3)$ so this is the IE. We could possibly solve this with some messy partial fractions but we won't.