MATH 246: Chapter 1 Section 2
Justin Wyss-Gallifent

1. Linear first-order ODEs.

   Recall that these will all have the form $p(t)y' + q(t)y = r(t)$ where p, q, r can be any functions of t.

   Example: $4y' + 5y = 0$

   Example: $4ty' + e^t y = \sin t$

2. We'll usually divide through by $p(t)$ to get these into what's called *linear normal form.* We'll relabel a bit and now assume they look like $y' + a(t)y = f(t)$ for functions a and f.

   These we can actually handle, and most of you did in Calculus II though it may be rusty. If we let $A(t)$ be an antiderivative of $a(t)$ so that $A'(t) = a(t)$ then observe:

$$\begin{aligned} y' + a(t)y &= f(t) \\ e^{A(t)}y' + e^{A(t)}a(t)y &= f(t)e^{A(t)} \\ \frac{d}{dt}\left(e^{A(t)}y\right) &= f(t)e^{A(t)} \\ e^{A(t)}y &= \int f(t)e^{A(t)}\ dt \\ y &= e^{-A(t)}\int f(t)e^{A(t)}\ dt \end{aligned}$$

   The only step that might concern you here is from line 2 to line 3. This is just the reverse of the product rule with a bit of chain rule thrown in. Reading it from line 3 to line 2 might be easier.

   This process can either be repeated for each problem or treated simply as a recipe.

   Be careful though, the $e^{-A(t)}$ is multiplied by the entire integral, meaning the $+C$ too when you integrate.

   Example: Consider $y' + 5y = 2$. We see that $a(t) = 5$ so $A(t) = 5t$ and the solution is

$$\begin{aligned} y &= e^{-5t}\int 2e^{5t}\ dt \\ &= e^{-5t}\left(\frac{2}{5}e^{5t} + C\right) \\ &= \frac{2}{5} + Ce^{-5t} \end{aligned}$$

   Example: Consider $ty' + 2y = t^4$ with $t > 0$. This is not in linear normal form so we divide by t to get $y' + \frac{2}{t}y = t^3$. Then $a(t) = \frac{2}{t}$ so $A(t) = 2\ln t$ and the solution is

$$\begin{aligned} y &= e^{-2\ln t}\int t^3 e^{2\ln t}\ dt \\ &= t^{-2}\int t^5\ dt \\ &= t^{-2}\left(\frac{1}{6}t^6 + C\right) \\ &= \frac{1}{6}t^4 + \frac{C}{t^2} \end{aligned}$$

Here's one with an IVP:

Example: Consider $y' - 6y = e^t$ with $y(0) = 2$. We see that $a(t) = -6$ so $A(t) = -6t$ and the general solution is

$$\begin{aligned} y &= e^{-(-6t)} \int e^t e^{-6t}\ dt \\ &= e^{6t} \int e^{-5t}\ dt \\ &= e^{6t} \left(-\frac{1}{5} e^{-5t} + C \right) \\ &= -\frac{1}{5} e^t + Ce^{6t} \end{aligned}$$

At this point $y(0) = -\frac{1}{5}e^0 + Ce^0 = -\frac{1}{5} + C = 2$ so that $C = \frac{11}{5}$ so the specific solution is

$$y = -\frac{1}{5} e^t + \frac{11}{5} e^{6t}$$

3. At this point you can probably see that solving a first-order linear ODE is as easy (or as hard) as first finding $A(t)$ and then finding $\int f(t)e^{A(t)}\ dt$.

4. Theory!

   By the FTOC if both $f(t)$ and $a(t)$ are continous on on an interval then not only will $A(t)$ exists but so will $e^{-A(t)}$ and $\int f(t)e^{A(t)}$. This means that if we have an initial value $y(t_I) = y_I$ then the interval of existence of the solution will be the largest open interval containing t_I on which both $f(t)$ and $a(t)$ are continouous. As before this lets us find the IE even when we can't solve the IVP.

   Example: Consider $y' + \frac{1}{t}y = \frac{1}{t-5}$ with $y(2) = 17$. Here $a(t) = \frac{1}{t}$ and $f(t) = \frac{1}{t-5}$. The largest open interval containing $t_I = 2$ on which both are continuous is $(0, 5)$ so this is the IE of the solution. Finding the solution is a different matter entirely, but it exists on $(0, 5)$!