

**MATH 246: Chapter 1 Section 3**  
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1. Separable ODEs.

A DE is *separable* if it can be written in the form  $y' = f(t)g(y)$ . The word separable comes from the fact that the right side is separated into a product of a function of  $t$  and a function of  $y$ .

**Example:**  $y' = ty$  is separable - it is already separated!

**Example:**  $ty' + y' = y^2$  is separable because it can be separated, first by factoring  $y'(t+1) = y^2$  and then dividing  $y' = \frac{y^2}{t+1}$  and thinking of it as  $y' = \left(\frac{1}{t+1}\right)y^2$ .

2. The method of solving a separable DE for the non-constant solutions is really slick:

$$\begin{aligned}y' &= f(t)g(y) \\ \frac{dy}{dt} &= f(t)g(y) \\ \frac{1}{g(y)}dy &= f(t)dt \\ \int \frac{1}{g(y)}dy &= \int f(t)dt\end{aligned}$$

Where the integral on the left is with respect to  $y$  and the integral on the right is with respect to  $t$ . Since both indefinite integrals should get their own constant, instead we just put a single  $+C$  on the right.

The  $\frac{1}{g(y)}$  looks really icky to integrate but in our examples it generally works out pretty nicely.

**Example:** Solve  $y' = \frac{t}{y+1}$ . We work as follows:

$$\begin{aligned}\frac{dy}{dt} &= \frac{t}{y+1} \\ (y+1)dy &= tdt \\ \int y+1 dy &= \int t dt \\ y^2 + y &= \frac{1}{2}t^2 + C\end{aligned}$$

3. Constant Solutions:

Notice that we divided by  $g(y)$ . What if  $g(y) = 0$ ? A really important thing to notice with separable ODEs is that if there are some  $y$  with  $g(y) = 0$  then those values of  $y$ , taken as functions, are solutions. This is because for those functions the derivative will be zero and the ODE will be satisfied. They are called the *constant solutions* to the ODE.

**Example:** Consider  $y' = (y^2 - 4)t^2$ . Here  $g(y) = y^2 - 4$  will equal 0 when  $y = \pm 2$ . Thus  $y = 2$  and  $y = -2$  are constant solutions (actual functions, which are constants and solutions) to the ODE.

#### 4. Implicit versus Explicit Solutions:

At this point you might notice something interesting. We have not actually obtained a function  $y = \dots$  and it may not be possible to. However we say that  $y$  is *implicitly* defined by the solutions and that we have found the *implicit* solution to the DE. In a case where we can actually solve for  $y$  then we say we've got the *explicit* solution. Here's one:

**Example:** Solve  $y' = t^2 y^2 + y^2$ . We work as follows:

$$\begin{aligned}\frac{dy}{dt} &= y^2(t^2 + 1) \\ y^{-2} dy &= t^2 + 1 dt \\ \int y^{-2} dy &= \int t^2 + 1 dt \\ -y^{-1} &= \frac{1}{3}t^3 + t + C \\ y &= \frac{-1}{\frac{1}{3}t^3 + t + C}\end{aligned}$$

Note there is the additional constant solution  $y = 0$ .

#### 5. Autonomous ODEs:

There is a special kind of separable ODE called *autonomous*. This occurs when  $f(t) = 1$  and so instead we have  $y' = g(y)$ . This can be solved like any other separable ODE. We only mention it because these will arise repeatedly over the course in various places.

**Example:** Solve  $y' = (y - 4)^2$ . Here  $g(y) = (y - 4)^2$  which equals 0 when  $y = 4$  so this is the constant solution. The nonconstant solutions we obtain as follows:

$$\begin{aligned}\frac{dy}{dt} &= (y - 4)^2 \\ (y - 4)^{-2} dy &= 1 dt \\ \int (y - 4)^{-2} dy &= \int 1 dt \\ -(y - 4)^{-1} &= t + C \\ (y - 4)^{-1} &= -(t + C) \\ (y - 4) &= \frac{-1}{t + C} \\ y &= \frac{-1}{t + C} + 4\end{aligned}$$

6. Two Small Notes:

- (a) The effect of initial values:

When we solve a separable ODE and get an implicit solutions for which there seems to be more than one explicit solution, an initial value usually tells us which one it is:

**Example:** Solve  $y' = \frac{t}{y}$  with  $y(1) = -3$ . First we solve the DE: r

$$\begin{aligned}\frac{dy}{dt} &= \frac{t}{y} \\ y \, dy &= t \, dt \\ \int y \, dy &= \int t \, dt \\ \frac{1}{2}y^2 &= \frac{1}{2}t^2 + C \\ y^2 &= t^2 + 2C \\ y &= \pm\sqrt{t^2 + 2C}\end{aligned}$$

Then  $y(1) = \pm\sqrt{1^2 + 2C} = -3$  so we are forced to use the negative in front of the square root. Thus  $-\sqrt{1^2 + 2C} = -3$  so  $1 + 2C = 9$  and  $C = 4$ . Then the explicit solution is  $y = -\sqrt{t^2 + 8}$ .

Note: There are no constant solutions here since  $g(y) = \frac{1}{y}$  is never 0.

- (b) Uniqueness (?) of solutions:

The existence of constant solutions often leads to non-unique solutions to IVPs. This can happen when a constant solution satisfies the DE but also the procedural method gives another solution. The way to manage this is to not forget to find your constant solutions and check if they satisfy the IV.

**Example:** Solve  $y' = y^{2/3}$  with  $y(0) = 0$ . Notice that  $y = 0$  is a constant solution which also satisfies the DE. However the DE is separable:

$$\begin{aligned}\frac{dy}{dt} &= y^{2/3} \\ y^{-2/3} \, dy &= 1 \, dt \\ \int y^{-2/3} \, dy &= \int 1 \, dt \\ 3y^{1/3} &= t + C \\ y &= \left(\frac{1}{3}t + \frac{1}{3}C\right)^3\end{aligned}$$

Then  $y(0) = \left(\frac{1}{3}C\right)^3 = 0$  so  $C = 0$ . This gives the additional solution  $y = \left(\frac{1}{3}t\right)^3 = \frac{1}{27}t^3$ .

7. Overlap

At this juncture it might be helpful to notice that an ODE doesn't need to be just one of the categories we've looked at - explicit, first-order linear, and separable - it could fall into more than one category.

**Example:**

$y' = ty$  is both separable and first-order linear.

**Example:**  $y' = t^2$  is all of explicit, separable and first-order linear.