

MATH 246: Chapter 1 Section 5

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The overarching goal of this section is to find things out about solutions to DEs without actually solving them explicitly. Instead we attack them graphically.

1. Phase Line Portraits for Autonomous Differential Equations.

- (a) As we've seen, autonomous differential equations look like $y' = g(y)$. Usually they have constant solutions when $g(y) = 0$. But what about when $g(y) \neq 0$?
- If $g(y) = 0$ then y is constant.
 - If $g(y) > 0$ then $y' > 0$ and y increases.
 - If $g(y) < 0$ then $y' < 0$ and y decreases.

Moreover as solutions move toward the constant solutions we see $g(y)$ tends toward 0 and so the graphs flatten out and become asymptotic.

What this means is that we can analyze the behavior of the solutions by looking at $g(y)$, specifically at the sign chart.

- (b) Stability. Moreover we'll see something happen near the constant solutions. Specifically one of three things can happen.
- i. If nearby solutions move away from the constant solution on both sides then the constant solution is unstable.
 - ii. If nearby solutions move toward the constant solution on both sides then the constant solution is stable.
 - iii. If there is different behavior on each side then the constant solution is *semistable*.

(c) Examples:

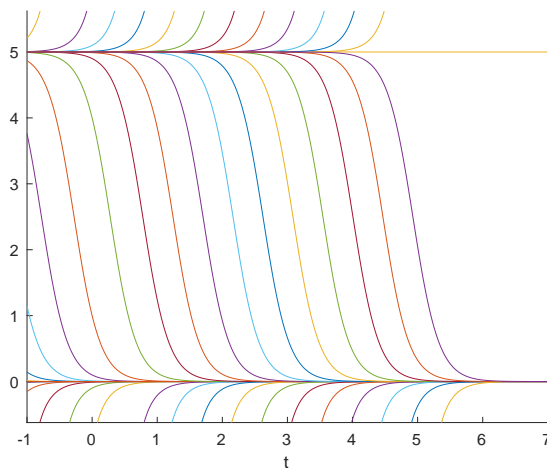
Example: Consider $y' = y(y - 5)$. Here is a sign chart for $g(y) = y(y - 5)$:

When y is		0		5		
$y' = g(y)$ is	+	0	-	0	+	
so $y(t)$ is	inc	cons	dec	const	inc	

This sign chart tells us that:

- There are constant solutions at $y = 0, 5$.
- Solutions having $y < 0$ will be increasing toward 0.
- Solutions having $0 < y < 5$ will be decreasing toward $y = 0$.
- Solutions having $5 < y$ will be increasing (toward infinity).

This suggests the following graph of the various *families* of solutions (drawn in Matlab):



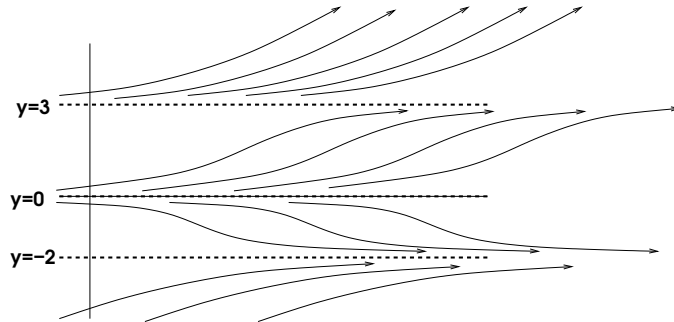
From these families of solutions we can draw all sorts of conclusions:

- The constant solution $y = 0$ is stable and the constant solution $y = 5$ is unstable.
- The particular solution $y(t)$ satisfying $y(0) = 1$ has $\lim_{t \rightarrow \infty} y(t) = 0$.
- The particular solution $y(t)$ satisfying $y(0) = 6$ has $\lim_{t \rightarrow \infty} y(t) = \infty$.
- If y is very close to 0 (either above or below) then over time it will tend towards 0. That is, $y = 0$ is stable.
- If y is very close to 5 (either above or below) then over time it will tend away from 0. That is, $y = 5$ is unstable.

Example: Consider $y' = y(y - 3)(y + 2)^2$. This has the following sign chart:

When y is		-2		0		3		
$y' = g(y)$ is	+	0	-	0	+	0	+	
so $y(t)$ is	inc	cons	dec	const	inc	const	inc	

And hence the following families of solutions (not drawn in Matlab because Matlab couldn't do a good job):



From these families of solutions we can draw all sorts of conclusions:

- The constant solution $y = -2$ is stable, the constant solution $y = 0$ is unstable, and the constant solution $y = 3$ is semistable.
- The particular solution $y(t)$ satisfying $y(0) = \alpha$ has $\lim_{t \rightarrow \infty} y(t) = 2$ when $-\infty < \alpha < 0$.
- The particular solution $y(t)$ satisfying $y(0) = \alpha$ is decreasing when $-2 < \alpha < 0$.
- $y = -2$ is stable.
- $y = 0$ is unstable.
- $y = 3$ is semistable.

2. Contour Plots of Implicit Solutions

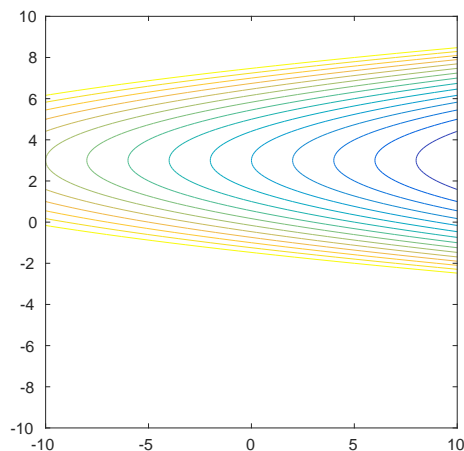
- (a) When we solve a separable DE we often get an implicit solution with a C in it. This implicit solution is an *equation*. If we pick various values of C and plot the resulting equations we get a *contour plot*.

What's useful about these contour plots is that the parts of the curves that form functions (y in terms of t) that satisfy the implicit solution. This means that we can pick a point on a curve and follow it as far left and right as possible and the result is the graph of an explicit solution to the DE.

Example: Consider $\frac{dy}{dx} = \frac{1}{2x-6}$. This is separable with general solution $y^2 - 6y = x + C$. This is not as bad as it looks:

$$\begin{aligned}y^2 - 6y &= x + C \\y^2 - 6y + 9 &= x + C + 9 \\(y - 3)^2 &= x + C + 9\end{aligned}$$

These are all parabolas opening right with their vertices at $y = 3$. If we sketch a few of these (note that they extend out forever, this is just a subset):

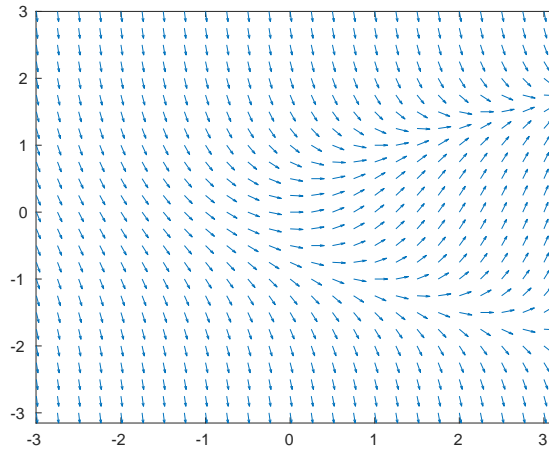


From this contour plot we can draw all sorts of conclusions:

- Solutions extend infinitely far to the right but not the left.
- The specific solution $y(t)$ satisfying $y(0) = 0$ is a decreasing function with $\lim_{t \rightarrow \infty} y(t) = -\infty$.
- Solutions are either always increasing or always decreasing.

3. Direction (Slope) Fields

- (a) As a last-ditch effort any $\frac{dy}{dt} = f(t, y)$ (any first-order) is essentially telling us the slope of a solution at a point. Consequently we can plug in lots of t and y and indicate what the slope would be of a solution passing through that point. The result is a *direction field* or *slope field*. Then we can trace functions which follow the field and draw conclusions.
- (b) **Example:** Here is the direction field for $\frac{dy}{dt} = t - y^2$, a hard DE to solve:



From this direction field we can draw all sorts of conclusions:

- We can trace the specific solution for which $y(-2) = 0$.
- We can trace a specific solution that has a relative minimum and we can suggest an initial value which would yield this solution.
- We can observe categories, for example not all solutions have relative minima.

Note however that we're somewhat restricted by the range we drew!

For class handout: Direction field for $\frac{dy}{dt} = t - y^2$:

