

MATH 246: Chapter 1 Section 6

Justin Wyss-Gallifent

The overarching goal of this section is to finally see some applications of DEs. We'll focus on three: Population dynamics, tanks, and motion. All three of these result in either first-order linear or separable differential equations which we can solve.

1. Population dynamics:

- (a) Introduction: In precalculus you probably learned that if a population grows at rate 5% then it obeys the formula

$$P = Ae^{0.05t}$$

But why? The answer is that to say "a population grows at rate 5%" means that the change in population at any time equals 5% of the actual population, meaning:

$$p' = 0.05p$$

This is a first order linear differential equation. If we rewrite it as $p' - 0.05p = 0$ then $a(t) = -0.05$ so $A(t) = -0.05t$ and the solution is:

$$p(t) = e^{-(-0.05t)} \int 0 dt = Ce^{0.05t}$$

That's why!

- (b) General Approach: Our general formula will involve a population with a certain growth rate R but in addition some new amount may arrive or depart every time period, maybe by being introduced, removed, etc. So in general we have

$$p' = Rp + a(t)$$

Our rate will always be constant but the amount that are introduced or subtracted may vary.

- (c) Examples:

Example: A population of monkeys starts with 100. It has a growth rate of 4% per year but an additional 8 monkeys join each year from a neighboring troop. Find the number of monkeys after t years.

Solution: Here we have $p' = 0.04p + 8$ with $p(0) = 100$.
The solution (work omitted) is $p = 300e^{0.04t} - 200$.

Example: In a certain neighborhood there is a mosquito problem. The population starts at 10M and has a growth rate of 20% monthly. Traps are put out and these traps kill 3M monthly. Find the number of mosquitos after t months and determine when the mosquitos will be wiped out.

Solution: Here we have $p' = 0.2p - 3$ with $p(0) = 10$.
The solution (work omitted) is $p = -5e^{0.2t} + 15$ and if we solve $-5e^{0.2t} + 15 = 0$ we get $t = 5 \ln(3) \approx 5.49$ months.

2. Tanks.

- (a) Introduction: We have a tank that contains a saltwater mixture. As time goes by, saltwater is being pumped in and out. Our goal is to know how much salt there is at any time t .
- (b) General Approach: If Q is the amount of salt at time t then we'll have

$$Q' = \text{Rate In} - \text{Rate Out}$$

The only confusing thing about these problems is we usually have to do some work with quantities to figure out the rates.

- (c) Examples:

Example: A tank initially contains 500L of saltwater with a concentration of 0.2kg/L. Saltwater with a concentration of 0.3kg/L is being pumped in at 10L/min while the tank is being emptied of the mixture at the same rate. Find the amount of salt in the tank at time t .

Solution: We're interested in the quantity of salt, not saltwater.

- Initially there is $(500 \text{ L})(0.2 \text{ kg/L})=100 \text{ kg}$ of salt so $Q(0) = 100$.
- Salt is entering at $(10 \text{ L/min})(0.3 \text{ kg/L})=3 \text{ kg/min}$.
- Salt is leaving at $(10 \text{ L/min})(Q \text{ kg} / 500 \text{ L})=0.02Q \text{ kg/min}$.

Note this is because at any instant there is y kg of salt in the tank and the tank always has 500 L of mixture in it because the rate in equals the rate out.

Therefore we have $Q' = 3 - 0.02Q$ with $Q(0) = 100$.

The solution (work omitted) is $Q = 150 - 50e^{-0.2t}$.

Example: A 300 gal tank initially contains 200 gal of saltwater with a concentration of 0.15 lb/gal. Saltwater with a concentration of 0.2 lb/gal is being pumped in at 6 gal/min while the tank is being emptied of the mixture at 4 gal/min. How much salt will be in the tank when it is full?

Solution: Observe that the tank does not start out full but gains 2 gal/min, meaning after time t it will have $200 + 2t$ gal in it. This will be important in the DE!

Again note:

- Initially there is $(200 \text{ gal})(0.15 \text{ lb/gal})=30 \text{ gal}$ of salt so $Q(0) = 30$.
- Salt is entering at $(6 \text{ gal/min})(0.2 \text{ lb/gal})=1.2 \text{ lb/min}$.
- Salt is leaving at $(4 \text{ gal/min})(Q \text{ lb} / 200 + 2t \text{ gal}) = \frac{4Q}{200+2t} \text{ lb/min}$.

Note this is because at any instant there is y lb of salt in the tank and the tank has $200 + 2t$ gal in it.

Therefore we have $Q' = 1.2 - \frac{4Q}{200+2t}$ with $Q(0) = 30$.

The solution (work omitted) is $Q = 0.4t - \frac{100000}{(t+100)^2} + 40$

3. Motion:

- (a) Introduction: In calculus you probably learned that a falling object with no air resistance has

$$a(t) = -9.8$$

But why? The answer comes from equating two forces. If the object has acceleration $a(t)$ and mass m then the force on it is $ma(t)$. The force from gravity is $-9.8m$. When we equate these we get

$$ma(t) = -9.8m$$

and then we cancel the m .

- (b) Adding some air: When we add air resistance there are now two forces. First there is gravity pulling down and then drag (air resistance for example) pushing up. These two forces combine to form the total force. We know the total force is $ma(t) = mv'(t)$. Therefore

$$ma(t) = \text{force of gravity} + \text{drag force}$$

The force of gravity is $-9.8m$. The drag force is harder, it's mkv^2 where k is the drag coefficient. Thus we have

$$ma(t) = -9.8m + mkv^2$$

or, cancelling the m again:

$$a(t) = -9.8 + kv^2$$

Finally we replace $a(t)$ by $v'(t)$ to get:

$$\frac{dv}{dt} = -9.8 + kv^2$$

- (c) General Approach: We'll generally just use the IVP

$$\frac{dv}{dt} = -9.8 + kv^2 \text{ with } v(0) = 0 \text{ (usually)}$$

to find v at time t and answer questions from that. Things to note:

- $v(0)$ might not be 0 if there is some initial velocity.
- In the Metric system k will be in m^{-1} , mass will be in kg and 9.8 stays as-is.
- In the English system k will be in ft^{-1} , mass will be in slugs and we use 32.2 (instead of 9.8).
- Terminal velocity occurs when $v' = 0$ so this is when $v = \sqrt{\frac{9.8}{k}}$.
- If we know v then we can also find out distance travelled since $v = h'$ and so $h(t_E) - h(t_I) = \int_{t_I}^{t_E} v(t) dt$
- We can certainly change from air to some other substance or from earth gravity to some other standard. Information would have to be given.
- It's insanely useful to know that

$$\int \frac{1}{x^2 - a^2} dx = -\frac{1}{a} \tanh^{-1} \frac{x}{a} + C \text{ where } \tanh(z) = \frac{e^{2z} - 1}{e^{2z} + 1} \text{ and } \tanh^{-1}(z) = \frac{1}{2} \ln \left(\frac{1+z}{1-z} \right)$$

(d) Examples:

Example: A skydiver leaps out of a plane at 3000m. The drag coefficient is $0.002m^{-1}$. What is the IVP here? What is her terminal velocity? Find her velocity at time t .

Solution: We have $\frac{dv}{dt} = -9.8 + 0.002v^2$ with $v(0) = 0$.

Her terminal velocity is $v = \sqrt{\frac{9.8}{0.002}} = \sqrt{4900} = 70$ m/s.

The solution to the IVP is shown here:

$$\begin{aligned}\frac{dv}{dt} &= -9.8 + 0.002v^2 \\ \frac{dv}{dt} &= 0.002(-4900 + v^2) \\ \frac{1}{v^2 - 4900} dv &= 0.002 dt \\ \int \frac{1}{v^2 - 4900} dv &= \int 0.002 dt \\ -\frac{1}{70} \tanh^{-1}\left(\frac{v}{70}\right) &= 0.002t + C \\ \tanh^{-1}\left(\frac{v}{70}\right) &= -0.14t + C \\ \frac{v}{70} &= \tanh(-0.14t + C) \\ \frac{v}{70} &= \tanh(-0.14t + C) \\ v &= 70 \tanh(-0.14t + C)\end{aligned}$$

Then $v(0) = 70 \tanh(C) = 0$ so $C = \tanh^{-1} 0 = 0$ and our final answer is

$$v = -70 \tanh(0.14t)$$