MATH 246: Chapter 1 Section 8
Justin Wyss-Gallifent

1. **A Bit of History and Introduction:** Suppose $H(x, y)$ is a function and y is a function of x. Then by the chain rule we know $\frac{d}{dx}H(x, y) = H_x(x, y) + H_y(x, y)\frac{dy}{dx}$.

   So now consider the following differential equation:

$$3x^2y^2 + 2x^3y\frac{dy}{dx} = 0$$

   You may notice that the left side looks like the result of the chain rule and is actually so, when $H(x, y) = x^3y^2$. Don't worry about if there's a formal method for where $H(x, y)$ comes from for now, just notice that $H_x(x, y) = 3x^2y^2$ and $H_y(x, y) = 2x^3y$. What this means is that the differential equation may be rewritten by undoing the chain rule on the left:

$$\underbrace{3x^2y^2 + 2x^3y\frac{dy}{dx}}_{\frac{d}{dx}[x^3y^2]} = 0$$

   So then when the derivative of something is zero, that thing is a constant:

$$\begin{aligned} \frac{d}{dx}\left[x^3y^2\right] &= 0 \\ x^3y^2 &= C \end{aligned}$$

   and we've solved it, at least implicitly!

2. **Definition and Method:** A differential equation is *exact* if it has the form:

$$H_x(x, y) + H_y(x, y)\frac{dy}{dx} = 0$$

   for some function $H(x, y)$. When a differential equation is exact, solving implicitly is as easy as finding $H(x, y)$ and setting $H(x, y) = C$ for any constant.

   Here are a few exact differential equations. For each, $H(x, y)$ is written in the middle and the implicit solution to the right.

Exact DE	$H(x, y)$	Solution to DE
$y + x\frac{dy}{dx} = 0$	$H(x, y) = xy$	$xy = C$
$y + (x + 2y)\frac{dy}{dx} = 0$	$H(x, y) = xy + y^2$	$xy + y^2 = C$
$\frac{1}{y} - \frac{x}{y^2}\frac{dy}{dx} = 0$	$H(x, y) = \frac{x}{y}$	$\frac{x}{y} = C$
$y\cos(xy) + x\cos(xy)\frac{dy}{dx} = 0$	$H(x, y) = \sin(xy)$	$\sin(xy) = C$

3. **Detecting Exactness and Finding H:** There is a trick to detecting whether a differential equation is exact. If the differential equation has the form:

$$M + N\frac{dy}{dx} = 0$$

then it is exact if and only if $M_y = N_x$. You can test all the ones above. Then you can check that this next one is not exact:

$$xy + y\frac{dy}{dx} = 0$$

In this case $M_y = x$ and $N_x = 0$. Not equal, not exact.

Once you know that your differential equation is exact, often you can guess at $H(x, y)$. However if you're struggling, there's a systematic method for finding it. Here's an example from above:

$$y + (x + 2y)\frac{dy}{dx} = 0$$

We want $H(x, y)$ with (A) $H_x(x, y) = y$ and (B) $H_y(x, y) = x + 2y$. Observe:

We want (A):	$H_x(x, y) = y$
This tells us that:	$H(x, y) = xy + h(y)$
From this line:	$H_y(x, y) = x + h'(y)$
But from (B):	$H_y(x, y) = x + 2y$
Set these equal:	$x + h'(y) = x + 2y$
Solve for $h'(y)$:	$h'(y) = 2y$
Find $h(y)$:	$h(y) = y^2 + D$
Put back into second line:	$H(x, y) = xy + y^2 + D$

We can choose any D so choose $D = 0$ to get $H(x, y) = xy + y^2$.

Example: Find $H(x, y)$ to solve $x + 1 + \frac{1}{y} - \frac{x}{y^2}\frac{dy}{dx} = 0$. Follow the exact procedure above, here we want (A) $H_x(x, y) = x + 1 + \frac{1}{y}$ and (B) $H_y(x, y) = -\frac{x}{y^2}$:

We want (A):	$H_x(x, y) = x + 1 + \frac{1}{y}$
This tells us that:	$H(x, y) = \frac{1}{2}x^2 + x + \frac{x}{y} + h(y)$
From this line:	$H_y(x, y) = -\frac{x}{y^2} + h'(y)$
But from (B):	$H_y(x, y) = -\frac{x}{y^2}$
Set these equal:	$-\frac{x}{y^2} + h'(y) = -\frac{x}{y^2}$
Solve for $h'(y)$:	$h'(y) = 0$
Find $h(y)$:	$h(y) = D$
Put back into second line:	$H(x, y) = \frac{1}{2}x^2 + x + \frac{x}{y} + D$

Then choose $D = 0$ to get $H(x, y) = \frac{1}{2}x^2 + x + \frac{x}{y}$ and the solution to our DE is $\frac{1}{2}x^2 + x + \frac{x}{y} = C$.

4. **Almost Exact:** It's not uncommon to have a differential equation which is not quite exact but can be made exact by multiplying through by some function called an *integrating factor*. For example the differential equation

$$2y + x\frac{dy}{dx} = 0$$

is not exact because $M_y = 2$ and $N_x = 1$ so $M_y \neq N_x$. But if we multiply through by x we get the new differential equation

$$2xy + x^2\frac{dy}{dx} = 0$$

which is exact because $M_y = 2x$ and $N_x = 2x$. Now $H(x, y) = x^2y$ and the solution is $x^2y = C$.

The question is how to come up with this integating factor. This can be challenging but we'll look at two simple cases. The key is that we've got our non-exact differential equation

$$M + N\frac{dy}{dx} = 0$$

and we wish to multiply through by some $\mu(x, y)$ such that the new differential equation

$$M\mu + N\mu\frac{dy}{dx} = 0$$

is exact. To be exact we'd need

$$\begin{aligned}(M\mu)_y &= (N\mu)_x \\ M_y\mu + M\mu_y &= N_x\mu + N\mu_x\end{aligned}$$

While this seems tricky (it's actually a partial differential equation!) we will only encounter the special cases when μ is a function of just x or just y.

The key is to take the above equation and say:

- If μ is a function of only x then $\mu_y = 0$. Rewriting this equation, can we see a $\mu(x)$ which would make this equation true?
- If μ is a function of only y then $\mu_x = 0$. Rewriting this equation, can we see a $\mu(y)$ which would make this equation true?

Note: We will only look at examples where μ is either a function of only x or only y and where μ is easy to figure out visually. Going beyond this can get seriously difficult.

5. **Examples:**

Example 1: Consider the non-exact diffential equation we've seen before:

$$2y + x\frac{dy}{dx} = 0$$

Here $M = 2y$ and $N = x$. We'd like:

$$\begin{aligned} M_y\mu + M\mu_y &= N_x\mu + N\mu_x \\ 2\mu + 2y\mu_y &= 1\mu + x\mu_x \end{aligned}$$

If $\mu = \mu(x)$ then $\mu_y = 0$ and this becomes:

$$\begin{aligned} 2\mu &= \mu + x\mu_x \\ x\mu_x &= \mu \\ \mu_x &= \frac{\mu}{x} \end{aligned}$$

We can see that $\mu(x) = x$ does the job. This is then our integrating factor and we multiply our original differential equation through by it to get the exact differential equation

$$2xy + x^2\frac{dy}{dx} = 0$$

which has $H(x, y) = x^2y$ and solution $x^2y = C$.

Example 2: Consider the non-exact differential equation

$$y + (x + xy)\frac{dy}{dx} = 0$$

Here $M = y$ and $N = x + xy$. We'd like:

$$\begin{aligned} M_y\mu + M\mu_y &= N_x\mu + N\mu_x \\ 1\mu + y\mu_y &= (1 + y)\mu + (x + xy)\mu_x \end{aligned}$$

If $\mu = \mu(y)$ then $\mu_x = 0$ and this becomes:

$$\begin{aligned} 1\mu + y\mu_y &= (1 + y)\mu \\ \mu + y\mu_y &= \mu + y\mu \\ \mu_y &= \mu \end{aligned}$$

We can see that $\mu(y) = e^y$ does the job. This is then our integrating factor and we multiple our original differential equation through by it to get the exact differential equation:

$$ye^y + (xe^y + xye^y)\frac{dy}{dx} = 0$$

This has $H(x, y) = xye^y$ and solution $xye^y = C$.