MATH 246: Chapter 2 Section 2
Justin Wyss-Gallifent

1. **Introduction:** Since even linear higher-order DEs are difficult we are going to simplify even more. For today we're going to look at *homogeneous* higher-order linear DEs, in which the forcing function $f(t)$ is equal to 0. That is:

First-Order	$y' + a(t)y = 0$
Second-Order	$y'' + a(t)y' + b(t)y = 0$
Third-Order	$y''' + a(t)y'' + b(t)y' + c(t)y = 0$
Etc.	Etc.

2. **A Motivational Example:** Consider the second-order homogeneous linear DE:

$$y'' - y' - 2y = 0$$

Next look at the two functions, don't worry about where they came from:

$$Y_1(t) = e^{2t} \text{ and } Y_2(t) = e^{-t}$$

We can easily see that these are both solutions to the DE by plugging them (and their derivatives) in and checking.

   (a) **Observation 1 - Getting More Solutions:**
   Notice that if we take a *linear combination* of these two, meaning

$$Y(t) = c_1 e^{2t} + c_2 e^{-t}$$

   where c_1 and c_2 are constants. Then we can easily see that this is also a solution to the DE by plugging it (and its derivatives) in and checking.

   (b) **Observation 2 - Getting All Solutions:**
   We can build new solutions from these two but can we build all solutions this way? Well suppose that we had some solution to the DE, call it $Y(t)$. What we want to know is if we can find c_1 and c_2 so that $Y(t) = c_1 e^{2t} + c_2 e^{-t}$ for this $Y(t)$?
   Well, suppose we find that $Y(0) = y_0$ and $Y'(0) = y_1$. Since $Y'(t) = 2c_1 e^{2t} - c_2 e^{-t}$ we would need

$$y_0 = Y(0) = c_1 + c_2$$
$$y_1 = Y'(0) = 2c_2 - c_2$$

   Can we find such values? Since $\det \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} = 0$ there is a unique solution.
   Notice now that since this is a solution to the IVP and since there is only one solution to the IVP this must be the solution we were looking for.

(c) **Observation 3 - Anything Special About Those Two?**

We can't just start with any two solutions. To see this observe that if we'd started with $Y_1(t) = e^{2t}$ and $Y_2(t) = 17e^{2t}$ that both of these are solutions. Again any linear combination $Y(t) = c_1e^{2t} + c_2 17e^{2t}$ is a solution. However is every solution to the DE a linear combination? Again, suppose $Y(t)$ is a solution and $Y(0) = y_0$ and $Y'(0) = y_1$. Then $Y'(t) = 2c_1e^{2t} + 34c_2e^{2t}$ and we would need

$$y_0 = Y(0) = c_1 + 17c_2$$
$$y_1 = Y'(0) = 2c_1 + 34c_2$$

Since $\det \begin{bmatrix} 1 & 17 \\ 2 & 34 \end{bmatrix} = 0$ there may be no solution. That is, we can't guarantee a solution.

3. **Theory:**

(a) **Theory for Second-Order** $y'' + a(t)y' + b(t)y = 0$

- For a second-order homogeneous linear DE we need to find two solutions $Y_1(t)$ and $Y_2(t)$ with a special relationship. That relationship is that their *Wronskian* does not equal the zero function, where:

$$W[Y_1, Y_2] = \det \begin{bmatrix} Y_1 & Y_2 \\ Y_1' & Y_2' \end{bmatrix}$$

  Alternately the two solutions cannot be multiples of each other. They form a *fundamental set* or *fundamental pair* of solutions.
- Every solution is then a linear combination of the fundamental pair. This means the general solution is $Y(t) = c_1Y_1(t) + c_2Y_2(t)$.
- A second-order IVP must provide $y(t_I)$ and $y'(t_I)$ in order to find the specific solution.
- This solution is unique on the *interval of existence* which is the largest open interval on which $a(t)$ and $b(t)$ are differentiable.

(b) **Theory for Third-Order** $y''' + a(t)y'' + b(t)y' + c(t)y = 0$

- For a third-order homogeneous linear DE we need to find three solutions $Y_1(t)$, $Y_2(t)$, and $Y_3(t)$ with a special relationship. That relationship is that their *Wronskian* does not equal the zero function, where:

$$W[Y_1, Y_2, Y_3] = \det \begin{bmatrix} Y_1 & Y_2 & Y_3 \\ Y_1' & Y_2' & Y_3' \\ Y_1'' & Y_2'' & Y_3'' \end{bmatrix}$$

  Alternately it must be impossible to write one of the solutions as a linear combination of the others. They form a *fundamental set* of solutions.
- Every solution is then a linear combination of the fundamental set. This means the general solution is $Y(t) = c_1Y_1(t) + c_2Y_2(t) + c_3Y_3(t)$.
- A third-order IVP must provide $y(t_I)$, $y'(t_I)$, and $y''(t_I)$ in order to find the specific solution.
- This solution is unique on the *interval of existence* which is the largest open interval on which $a(t)$ and $b(t)$ and $c(t)$ are differentiable.

(c) **Theory for Higher-Order:**

You can probably see the pattern.

4. **Practice for Both:**

Here are some examples:

Example: Consider $y'' + 4y = 0$. First we'll show that $Y_1(t) = \sin(2t)$ and $Y_2(t) = \cos(2t)$ form a fundamental pair. We check they are solutions (omitted) and we check:

$$W[Y_1, Y_2] = \det \begin{bmatrix} \sin(2t) & \cos(2t) \\ 2\cos(2t) & -2\sin(2t) \end{bmatrix} = -2\sin^2(2t) - 2\cos^2(2t) = -2 \not\equiv 0$$

This tells us that $Y_1(t)$ and $Y_2(t)$ form a fundamental pair and that the general solution is:

$$Y(t) = c_1 \sin(2t) + c_2 \cos(2t)$$

So now if we have the IVP with $Y(0) = 4$ and $Y'(0) = 2$ we can find the specific solution by first finding:

$$Y'(t) = 2c_1 \cos(2t) - 2c_2 \sin(2t)$$

and then solving the system:

$$\begin{aligned} 4 = Y(0) &= c_1 \sin(2(0)) + c_2 \cos(2(0)) = c_2 \\ 2 = Y'(0) &= 2c_1 \cos(2(0)) - 2c_2 \sin(2(0)) = 2c_1 \end{aligned}$$

So that $c_1 = 1$ and $c_2 = 4$ and the specific solution is:

$$Y(t) = \sin(2t) + 4\cos(2t)$$

Example: Consider $(1 + t^2)y'' - 2ty' + 2y = 0$. First we'll show that $Y_1(t) = t$ and $Y_2(t) = t^2 - 1$ form a fundamental pair. We check they are solutions (omitted) and we check:

$$W[Y_1, Y_2] = \det \begin{bmatrix} t & t^2 - 1 \\ 1 & 2t \end{bmatrix} = 2t^2 - (t^2 - 1) = t^2 + 1 \not\equiv 0$$

This tells us that $Y_1(t)$ and $Y_2(t)$ form a fundamental pair and that the general solution is:

$$Y(t) = c_1 t + c_2(t^2 - 1)$$

So now if we have the IVP with $Y(2) = -5$ and $Y'(2) = 7$ we can find the specific solution by first finding:

$$Y'(t) = c_1 + 2c_2 t$$

and then solving the system:

$$\begin{aligned} -5 = Y(2) &= c_1(2) + c_2(2^2 - 1) = 2c_1 + 3c_2 \\ 7 = Y'(2) &= c_1 + 2c_2(2) = c_1 + 4c_2 \end{aligned}$$

So that $c_1 = -\frac{41}{5}$ and $c_2 = -\frac{19}{5}$ and the specific solution is:

$$Y(t) = -\frac{41}{5}t + \frac{19}{5}(t^2 - 1)$$

Example: Consider $D^3y - 2D^2y = 0$ First we'll show that $Y_1(t) = 1$, $Y_2(t) = t$ and $Y_3(t) = e^{2t}$ form a fundamental set. We check they are solutions (omitted) and we check:

$$W[Y_1, Y_2, Y_3] = \det \begin{bmatrix} 1 & t & e^{2t} \\ 0 & 1 & 2e^{2t} \\ 0 & 0 & 4e^{2t} \end{bmatrix} = 4e^{2t} \not\equiv 0$$

This tells us that $Y_1(t)$, $Y_2(t)$ and $Y_3(t)$ form a fundamental set and that the general solution is:

$$Y(t) = c_1 + c_2 t + c_3 e^{2t}$$

So now if we have the IVP with $Y(0) = 1$, $Y'(0) = 0$ and $Y''(0) = -4$ we can find the specific solution by first finding:

$$\begin{aligned} Y'(t) &= c_2 + 2c_3 e^{2t} \\ Y''(t) &= 4c_3 e^{2t} \end{aligned}$$

and then solving the system:

$$\begin{aligned} 1 &= Y(0) = c_1 + c_3 \\ 0 &= Y'(0) = c_2 + 2c_3 \\ -4 &= Y''(0) = 4c_3 \end{aligned}$$

So that $c_3 = -1$, $c_2 = 2$ and $c_1 = 2$ and the specific solution is:

$$Y(t) = 2 + 2t - e^{2t}$$

5. **More about Fundamental Sets:**

(a) **Natural Fundamental Sets (OMITTED FOR NOW)**

There's more than just one fundamental set, and one that comes up a lot is called the *natural fundamental set.*

In the second-order case this is the set $\{Y_1, Y_2\}$ with Y_1 having $Y_1(t_I) = 1$ and $Y_1'(t_I) = 0$ and with Y_2 having $Y_2(t_I) = 0$ and $Y_2'(t_I) = 1$.

In the third-order case this is the set $\{Y_1, Y_2, Y_3\}$ with Y_1 having $Y_1(t_I) = 1$, $Y_1'(t_I) = 0$, and $Y_1''(t_I) = 0$, with Y_2 having $Y_2(t_I) = 0$, $Y_2'(t_I) = 1$, and $Y_2''(t_I) = 0$, and with Y_3 having $Y_3(t_I) = 1$, $Y_3'(t_I) = 0$, and $Y_3''(t_I) = 1$,

Beyond there you can probably see the pattern.

(b) **Reduction of Order (OMITTED)**

The big question of course is where the fundamental set comes from. We'll address that a bit later but for now we have one helper.

If we have one solution $Y_1(t)$ then the second one is very often a multiple of the first. So we can set $Y_2(t) = uY_1(t)$ and when we plug this into the DE and use the fact that $Y_1(t)$ is a solution we end up in a situation where we can find a first-order DE (hence the name) that we can use to find u.

Example: You can check that $Y_1(t) = e^{5t}$ is a solution to $y'' - 3y' - 10y = 0$. To find the other by reduction of order we put $Y_2(t) = ue^{5t}$. We then find

$$Y_2'(t) = u'e^{5t} + 5ue^{5t} \text{ and}$$
$$Y_2''(t) = u''e^{5t} + 5u'e^{5t} + 5u'e^{5t} + 25ue^{5t} = u''e^{5t} + 10u'e^{5t} + 25ue^{5t}$$

and plug these into the DE:

$$\begin{aligned} y'' - 3y' - 10y &= 0 \\ (u''e^{5t} + 10u'e^{5t} + 25ue^{5t}) - 3(u'e^{5t} + 5ue^{5t}) - 10(ue^{5t}) &= 0 \\ u'' + 10u' + 25u - 3u' - 15u - 10u &= 0 \\ u'' + 7u' &= 0 \end{aligned}$$

If we let $w = u'$ then this gives us $w' + 7w = 0$ which has solution $w = Ce^{-7t}$ and so $u' = Ce^{-7t}$ and so $u = -\frac{1}{7}Ce^{-7t} + D$ and another solution is

$$Y_2(t) = \left(-\frac{1}{7}Ce^{-7t} + D\right) e^{5t} = -\frac{1}{7}Ce^{-2t} + De^{5t}$$

Since this is true for any C and D we can pick the solution

$$Y_2(t) = e^{-2t}$$

for which $W[Y_1, Y_2] \not\equiv 0$ and we have our fundamental pair.