MATH 246: Chapter 2 Section 3
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1. **Introduction:**

   As the course progresses we'll start running into matrices and we'll need some basic facts. For now we simply need to know what a matrix is, what a determinant is, and what they can be used for.

2. **Matrices:**

   A matrix is basically a rectangular array of numbers. In this course pretty much all the matrices we'll work with will be square and either 2×2 or 3×3.

   Examples:

$$A = \begin{bmatrix} 3 & -2 \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 3 \\ -5 & 7 \end{bmatrix} \qquad C = \begin{bmatrix} 4 & 3 & 1 \\ -2 & 5 & 0 \\ 0 & 8 & -3 \end{bmatrix}$$

3. **Determinants:**

   The determinant of a matrix is a single number associated with the matrix which tells us certain properties of that matrix. It is the single most important number associated with a matrix. It can be denoted either by putting det in front of the matrix or by putting the matrix values inside vertical bars like absolute values.

   It can be defined recursively but we'll only need it for 2×2 and 3×3 so here are the rules:

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

and

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \cdot \det \begin{bmatrix} e & f \\ h & i \end{bmatrix} - b \cdot \det \begin{bmatrix} d & f \\ g & i \end{bmatrix} + c \cdot \det \begin{bmatrix} d & e \\ g & h \end{bmatrix}$$

Example: $\det \begin{bmatrix} 3 & -2 \\ 0 & 1 \end{bmatrix} = (3)(1) - (-2)(0) = 3$

Example: $\det \begin{bmatrix} 1 & 3 \\ -5 & 7 \end{bmatrix} = (1)(7) - (3)(-5) = 22$

Example:

$$\begin{aligned} \det \begin{bmatrix} 4 & 3 & 1 \\ -2 & 5 & 0 \\ 0 & 8 & -3 \end{bmatrix} &= 4 \det \begin{bmatrix} 5 & 0 \\ 8 & -3 \end{bmatrix} - 3 \det \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} + 1 \det \begin{bmatrix} -2 & 5 \\ 0 & 8 \end{bmatrix} \\ &= 4(-15) - 3(6) + 1(-16) \\ &= -94 \end{aligned}$$

4. **First Use:**

   In linear algebra matrics are used to solve linear systems of equations. We don't need to do that but we do need to know that determinants of matrices can tell us information about the solutions.

   (a) **The Basic Fact:**

   If we put the coefficients of the variables into a matrix and find the determinant then this determinant will be nonzero if and only if there is a unique solution to the system. If we do get zero then there will be either no solutions or infinitely many solutions. For now we need not distinguish between these outcomes.

   Example: The system

$$\begin{aligned} 2x + 3y &= 7 \\ 5x - 7y &= 23 \end{aligned}$$

   Since $\det \begin{bmatrix} 2 & 3 \\ 5 & -7 \end{bmatrix} = -29 \neq 0$ there is only one solution.

   Example: The system

$$\begin{aligned} 4x + 8y &= 3 \\ 6x + 12y &= -8 \end{aligned}$$

   Since $\det \begin{bmatrix} 4 & 8 \\ 6 & 12 \end{bmatrix} = 0$ there are either no solutions or infinitely many solutions.

   Example: The system

$$\begin{aligned} 4x + 3y + 1z &= 7 \\ -2x + 5y + 0z &= -17 \\ 0x + 8y - 3z &= 2 \end{aligned}$$

   Since $\det \begin{bmatrix} 4 & 3 & 1 \\ -2 & 5 & 0 \\ 0 & 8 & -3 \end{bmatrix} = -94 \neq 0$ there is only one solution.

(b) **Ramifications for Homogenous Systems:**

A homogeneous linear system is when all the constant terms are 0. Setting all the variables to be zero always gives a solution, called the *trivial solution*. In this case if the determinant is zero then there must be infinitely many solutions, meaning there are *nontrivial solutions*, and if the determinant is nonzero then there is only the trivial solution, so there are no nontrivial solutions.

Example: The system

$$\begin{aligned} 2x + 3y &= 0 \\ 5x - 7y &= 0 \end{aligned}$$

Has the trivial solution $x = y = 0$. In addition since $\det \begin{bmatrix} 2 & 3 \\ 5 & -7 \end{bmatrix} = -29 \neq 0$ that's the only solution.

Example: The system

$$\begin{aligned} 4x + 8y &= 0 \\ 6x + 12y &= 0 \end{aligned}$$

Has the trivial solution $x = y = 0$. In addition since $\det \begin{bmatrix} 4 & 8 \\ 6 & 12 \end{bmatrix} = 0$ there are other nontrivial solutions.