MATH 246: Chapter 2 Section 4
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1. **Introduction:**

   We've established the fact that for an n^{th} order homogeneous linear differential equation that we need to find a fundamental set of n solutions denoted $Y_1, ..., Y_n$. Once we do this we have the general solution
   $$Y(t) = c_1Y_1 + c_2Y_2 + ...c_nY_n$$
   The next question is how to get that fundamental set.

   This is hard, so we'll focus on the simpler situation where the DE has constant coefficients. Examples:
   $$\begin{gathered} y'' + 2y' - 3y = 0 \\ 2y''' - 5y = 0 \\ D^4y + D^3y - 2D^2y + Dy + y = 0 \end{gathered}$$

2. **Inspirational Example:** Consider $y'' + 2y' - 3y = 0$. Here because we've got constant multiples of derivatives added to get zero, we think that perhaps solutions might be functions whose derivatives are constant multiples of themselves. The primary example is e^{zt} for various z. So let's try - if $y = e^{zt}$ is a solution to our equation then $y' = ze^{zt}$ and $y'' = z^2e^{zt}$ and we have:
   $$\begin{aligned} y'' + 2y' - 3y &= 0 \\ z^2e^{zt} + 2ze^{zt} - 3e^{zt} &= 0 \\ (z^2 + 2z - 3)e^{zt} &= 0 \end{aligned}$$
   Since e^{zt} is always positive we must have
   $$\begin{aligned} z^2 + 2z - 3 &= 0 \\ (z + 3)(z - 1) &= 0 \end{aligned}$$
   and so either $z = -3$ or $z = 1$.

   Lo and behold we've actually found two solutions, and we only needed two. We've found
   $$Y_1(t) = e^{-3t} \text{ and } Y_2(t) = e^{1t}$$
   Thus the general solution is
   $$Y(t) = c_1e^{-3t} + c_2e^t$$

3. **General Idea:** We see that the DE $y'' + 2y' - 3y = 0$ gave us a polynomial $z^2 + 2z - 3 = 0$. This will happen in every case, that polynomial is called the *characteristic polynomial*. The roots of the polynomial will give us solutions and we will get enough complete our fundamental set. However there are nuances, primarily that we have to deal with both real and complex roots and each of these has two subcategories.

   In addition, though we won't prove this, the process is guaranteed to result in a fundamental set, meaning the Wronskian would be nonzero if we checked it.

4. **Real Simple Roots:** A *real simple root* is a root which only appears once when we factor the characteristic polynomial. For a simple real root $z = r$ we get a solution e^{rt}. If there are n simple real roots then we get n solutions and we're done.

   Example: The DE $y'' - 3y' - 10y = 0$ has characteristic polynomial $z^2 - 3z - 10$ or $(z-5)(z+2)$ with roots $z = 5$ and $z = -2$. So we have two simple real roots and therefore the fundamental set is $\{e^{5t}, e^{-2t}\}$ and the general solution is $Y(t) = c_1e^{5t} + c_2e^{-2t}$.

   Example: The DE $y''' - 5y'' + 6y' = 0$ has characteristic polynomial $z^3 - 5z^2 + 6z$ or $z(z-3)(z-2)$ with roots $z = 0$, $z = 3$ and $z = 2$. So we have three simple real roots and therefore the fundamental set is $\{1, e^{3t}, e^{2t}\}$ (notice that $e^{0t} = 1$) and the general solution is $Y(t) = c_1 + c_2e^{3t} + c_3e^{2t}$

5. **Real Multiple Roots** A *real multiple root* is a root which appears more than once when we factor the characteristic polynomial.

   Example: The DE $D^3y - 15D^2y + 75Dy - 125y = 0$ has characteristic polynomial $z^3 - 15z^2 + 75z - 125$ or $(z-5)^3$. there is only the root $z = 5$ with multiplicity $m = 3$.

   Example: The DE $y''' - 4y'' + 4y' = 0$ has characteristic polynomial $z^3 - 4z^2 + 4z$ or $z(z-2)^2$. There is the root $z = 0$ which is a simple root (multiplicity 1) and the root $z = 2$ with multiplicity $m = 2$.

   For a real multiple root $z = r$ with multiplicity m we get m solutions:

   $$e^{rt}, te^{rt}, ..., t^{m-1}e^{rt}$$

   We'll discuss where these come from in a later section.

   Example: The DE $D^3y - 15D^2y + 75Dy - 125y = 0$ has characteristic polynomial $z^3 - 15z^2 + 75z - 125$ or $(z-5)^3$. there is only the root $z = 5$ with multiplicity $m = 3$. So we have the fundamental set $\{e^{5t}, te^{5t}, t^2e^{5t}\}$ and the general solution $Y(t) = c_1e^{5t} + c_2te^{5t} + c_3t^2e^{5t}$.

   Example: The DE $y''' - 4y'' + 4y' = 0$ has characteristic polynomial $z^3 - 4z^2 + 4z$ or $z(z-2)^2$. There is the root $z = 0$ which is a simple root (multiplicity 1) and the root $z = 2$ with multiplicity $m = 2$. So we have the fundamental set $\{1, e^{2t}, te^{2t}\}$ and the general solution $Y(t) = c_1 + c_2e^{2t} + c_3e^{3t}$.

6. **Complex Simple Roots:** Complex roots always come in pairs. This means that if $r+si$ is a root then so is $r-si$. A *complex simple root pair* is a pair of roots $r \pm si$ which appears only once when we factor the characteristic polynomial. We may have to actually solve via the quadratic formula to really see what's up.

Example: The DE $y''+y'+2y=0$ has characteristic polynomial z^2+z+2. This doesn't obviously factor. To find the roots we set it equal to 0 and use the quadratic formula to get $z=\frac{-1\pm\sqrt{1^2-4(1)(2)}}{2(1)}=-\frac{1}{2}\pm\frac{\sqrt{-7}}{2}=-\frac{1}{2}\pm\frac{\sqrt{7}}{2}i$.

Example: The DE $y''+4y=0$ has characteristic polynomial z^2+4 and setting $z^2+4=0$ gives $z=\pm\sqrt{-4}=0\pm 2i$. We've made the 0 clear for a reason as you'll see.

To see what's going to happen, let's just charge ahead for a minute. If $r+si$ is a root then $e^{(r+si)t}$ is a solution. But

$$e^{(r+si)t}=e^{rt}e^{(st)i}=e^{rt}(\cos(st)+i\sin(st))$$

Since $r-si$ is also a root then $e^{(r-si)t}$ is also a solution and

$$e^{(r-si)t}=e^{rt}e^{(-st)i}=e^{rt}(\cos(-st)+i\sin(-st))=e^{rt}(\cos(st)-i\sin(st))$$

But these are complex solutions which don't fill out our fundamental set. But since linear combinations of solutions are solutions, we can be sneaky and observe:

- $\frac{1}{2}$(sum of complex solutions)$=e^{rt}\cos(st)$ is a solution.
- $\frac{1}{2i}$(difference of complex solutions)$=e^{rt}\sin(st)$ is a solution.

Thus for each complex simple root pair $z=r\pm si$ we get a pair of solutions $e^{rt}\cos(st)$ and $e^{rt}\sin(st)$.

Example: The DE $y''+y'+2y=0$ has characteristic polynomial z^2+z+2 with roots we saw as $-\frac{1}{2}\pm\frac{\sqrt{7}}{2}i$. The fundamental set is then $\left\{e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{7}}{2}t\right), e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{7}}{2}t\right)\right\}$ and the general solution is $Y(t)=c_1e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{7}}{2}t\right)+c_2e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{7}}{2}t\right)$.

Example: The DE $y''+4y=0$ has characteristic polynomial z^2+4 with roots $z=0\pm 2i$. The fundamental set is then $\{\cos(2t),\sin(2t)\}$ and the general solution is $Y(t)=c_1\cos(2t)+c_2\sin(2t)$.

Example: The DE $y'''-y''-4y'-6y=0$ has characteristic polynomial z^3-z^2-4z-6 or $(z-3)(z^2+2z+2)$. We see that $z=3$ is a root but for the rest we set $z^2+2z+2=0$ and get $z=\frac{-2\pm\sqrt{2^2-4(1)(2)}}{2}=-1\pm 1i$. The fundamental set is then $\{e^{3t},e^{-t}\cos(t),e^{-t}\sin(t)\}$ and the general solution is $Y(t)=c_1e^{3t}+c_2e^{-t}\cos(t)+c_3e^{-t}\sin(t)$.

7. **Complex Multiple Roots:** This expands like with real roots. For each complex multiple root pair $z = r \pm si$ with multiplicity m we get m pairs of solutions

$$e^{rt}\cos(st), te^{rt}\cos(st), ..., t^{m-1}e^{rt}\cos(st)$$

$$e^{rt}\sin(st), te^{rt}\sin(st), ..., t^{m-1}e^{rt}\sin(st)$$

Example: The DE $y'''''' + 8y''''' + 65y'''' + 232y''' + 904y'' + 1440y' + 3600y = 0$ has characteristic polynomial $z^6 + 8z^5 + 65z^4 + 232z^3 + 904z^2 + 1440z + 3600$ or $(z^2 + 9)(z^2 + 4z + 20)^2$. The first part gives us $z = 0 \pm 3i$ and the second part gives us $z = \frac{-4 \pm \sqrt{4^2 - 4(1)(20)}}{2} = -2 \pm 4i$ with multiplicity 2. The fundamental set is then $\{\cos(3t), \sin(3t), e^{-2t}\cos(4t), e^{-2t}\sin(4t), te^{-2t}\cos(4t), te^{-2t}\sin(4t)\}$ and the general solution is $Y(t) = c_1\cos(3t) + c_2\sin(3t) + c_3e^{-2t}\cos(4t) + c_4e^{-2t}\sin(4t) + c_5te^{-2t}\cos(4t) + c_6te^{-2t}\sin(4t)$.