MATH 246: Chapter 2 Section 5
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1. **Introduction:**

   Now that we know how to handle homogeneous linear differential equations with constant coefficents we'd like to see what happens if we make one small change. We'll look at nonhomogeneous linear differential equations. For now we'll momentarily even remove the requirement that we have constant coefficients. Examples:

$$y'' + 2y' - 3y = e^t$$
$$2y''' - 5ty = \cos(2t)$$
$$y'''' + e^t y''' - 2ty'' + Dy + t^2 y = t$$

2. **An Inspirational Example:**

   The general idea is actually quite simple. To see, let's look at the example

$$y'' + 2y' - 3y = f(t)$$

   where $f(t)$ is unknown but nonzero.

   Suppose that somehow we obtained just one solution. Call that solution $Y_p(t)$ where the p stands for *particular*. Suppose that $Y(t)$ is any other solution. Then look at what happens when we plug in $Y(t) - Y_p(t)$:

$$\begin{aligned}
(Y(t) - Y_p(t))'' + 2(Y(t) - Y_p(t))' - 3(Y(t) - Y_p(t)) \\
&= [Y''(t) + 2Y'(t) - 3Y(t)] - \left[Y_p''(t) + 2Y_p'(t) - 3Y_p(t)\right] \\
&= f(t) - f(t) \\
&= 0
\end{aligned}$$

   What this is telling us is that $Y(t) - Y_p(t)$ is a solution to the *homogeneous version* of the DE. Since we know that all the solutions to the homogeneous version look like $c_1 e^{-3t} + c_2 e^t$ this then tells us that

$$\begin{aligned}
Y(t) - Y_p(t) &= c_1 e^{-3t} + c_2 e^t \\
Y(t) &= Y_p(t) + c_1 e^{-3t} + c_2 e^t
\end{aligned}$$

3. **General Method:**

   What this tells us is actually pretty fantastic. It says that when we're confronted by an n^{th} order nonhomogeneous linear differential equation, all we need to do is two things:

   (a) Find the fundamental set $\{Y_1, ..., Y_n\}$ for the homogeneous version.

   (b) Find one single solution Y_p for the original differential equation.

   Then the general solution to the original differential equation is

$$Y(t) = Y_p(t) + c_1 Y_1(t) + ... + c_n Y_n(t)$$

   Of course these two things may not be easy. If the homogeneous version has constant coefficients at least part (a) is easy. Let's gloss over these for now to see things in practice.

4. **Examples:**

   Here are some examples of both DEs and IVPs:

Example: Consider the differential equation $y'' + 4y = 4t$.

(a) The homogeneous version $y'' + 4y = 0$ has fundamental set $\{\cos(2t), \sin(2t)\}$.

(b) The function $Y_p(t) = t$ is a solution to $y'' + 4y = 4t$.

Thus the general solution to $y'' + 4y = 4t$ is:

$$Y(t) = t + c_1 \cos(2t) + c_2 \sin(2t)$$

Example: Consider the differential equation $D^3y - 2D^2y = 9e^{3t}$.

(a) The homogeneous version $D^3y - 2D^2y = 0$ has fundamental set $\{1, t, e^{2t}\}$.

(b) The function $Y_p(t) = e^{3t}$ is a solution to $D^3y - 2D^2y = 9e^{3t}$.

Thus the general solution to $D^3y - 2D^2y = 9e^{3t}$ is:

$$Y(t) = e^{3t} + c_1 + c_2t + c_3e^{2t}$$

Example: Consider the initial value problem $(1+t^2)y'' - 2ty' + 2y = 6$ with $Y(0) = 2$ and $Y'(0) = 1$.

(a) The homogeneous version $(1+t^2)y'' - 2ty' + 2y = 0$ has fundamental set $\{t, t^2 - 1\}$.

(b) The function $Y_p(t) = 3$ is a solution to $(1+t^2)y'' - 2ty' + 2y = 6$.

Thus the general solution to $(1+t^2)y'' - 2ty' + 2y = 6$ is:

$$Y(t) = 3 + c_1t + c_2(t^2 - 1)$$

To solve the IVP we find

$$Y'(t) = c_1 + 2tc_2$$

and we solve:

$$\begin{aligned} 2 &= Y(0) = 3 + c_1(0) + c_2(0^2 - 1) \\ 1 &= Y'(0) = c_1 + 2(0)c_2 \end{aligned}$$

This tells us that $c_2 = 1$ and $c_1 = 1$ and so the specific solution is

$$Y(t) = 6 + t + (t^2 - 1)$$